Another application of group theory to music is *change-ringing*, which refers to the process whereby people playing church bells can ring the bells in every possible order. To see what I mean, suppose that there are 3 bells labelled 1,2,3. (It’s traditional to label the bell with the highest pitch as 1, the next highest 2, and so on. The practice of change-ringing developed in England, where there can be as many as 6, 7 or 8 bells in a church tower, all tuned to different pitches. Closer to hand, there is a peal of 10 bells installed at Grace Episcopal Church on Wentworth St.; the largest bell in the set weighs almost a ton. They are tuned to the notes of an E-flat major scale, plus an extra F and G at the top.)

For 3 bells, it turns out that there are 6 possible orders in which you could play all 3 bells:

- 123
- 132
- 213
- 231
- 312
- 321

In general, if you have n bells (labelled 1...n) there are n! different ways of playing them in some order. For example, for 4 bells there are 4!=24 orders, for 5 bells there are 5!=120, and so on. Harkelroad (Ch. 5) refers to a way of playing the bells in some order (with no repeats) as an *extent*.

Each of the bells is operated by pulling on a rope, which causes the bell to swing until it is nearly upside down, at which point the clapper inside the bell hits the bell and it rings. Meanwhile, the person operating the next bell in the sequence has raised that bell to the top of its swing, and it rings. When the bell is nearly upside down, a little tug on the rope adds enough momentum to ensure that, in a few seconds, it swings all the way in the other direction so that it’s near upside down again. From then on, the bell can be kept ringing with comparatively small tugs on the rope at the appropriate times.

So, each bell goes through a cycle of two rings, one on the handstroke and one on the backstroke. If the ropes are pulled in the right way,
there will be enough time between the two rings for all the other bells to ring. For this reason, each ordering of the bells is rung through twice, once on handstrokes and once on backstrokes. However, in describing patterns of change-ringing I won’t bother with writing down the sequence of rings only once rather than twice.
The fact that only small adjustments can be made in the timing of when each bell rings means that the position of a bell in the extent can only be changed by one place at a time. In other words, the physical constraints of bell-ringing imply that the only way we can change the order of bells is by swapping the positions of two bells that are one after the other. For example, with three bells the change 123 -> 312 is not allowed because the “3” bell moved more than 1 place; however, the change 123 -> 213 is allowed.

The mathematical study of bell-ringing comes down to the question: how can we run through all possible extents only by exchanging bells in adjacent positions? (Note that we are not allowed to repeat an extent until the end.) For example, here is a solution for three bells:

123 213 231 321 312 132 123,

where we alternate between swapping the bells in the first two positions and swapping the bells in the last two positions.

**Permutation Notation** Changing the order in which n objects are positioned is called making a *permutation* of those objects. Since we’re already using numbers to refer to the bells, we’ll follow Harkelroad (Chapter 5) in labelling the positions within an extent as A,B,C,D, etc. To denote operation of swapping the bells in positions A and B, we’ll write (AB). So, the above sequence of permutations on 3 bells could be written as

123 -(AB)-> 213 -(BC)-> 231 -(AB)-> 321 -(BC)-> 312 -(AB)-> 132 -(BC)-> 213

Recall that, when we multiply transformations, the rightmost one is the first one performed. So, the transformations we perform here can be written as
(AB), (BC)(AB), (AB)(BC)(AB), ((BC)(AB))^2, (AB) ((BC)(AB))^2, and finally ((BC)(AB))^3=() the identity permutation. (Note that no earlier transformation is the identity, and this implies that (BC) and (AB) do not commute. In fact, one way of understanding the relation ((BC)(AB))^3 = () is that the composition (BC)(AB) equals (ACB), the cyclic permutation that shifts all objects one place backward, wrapping around the end: 123 - (ACB) -> 231 Clearly, if this is done three times we get the identity.)

What about 4 bells? Here we could consider the operation of simultaneously swapping two adjacent bells (i.e., the transformation (AB)(CD)). Here’s the beginning of a bell-ringing pattern called *Plain Bob Minimus*:

1234 -> 2143 -> 2413 -> 4231 -> 4321 -> 3412 -> 3142 -> 1324 ...

where we have alternated between applying (AB)(CD) applying (BC). Notice that one more (BC) swap will bring us back to the beginning; in other words, ((BC) (AB) (CD))^4 = the identity.

But there are 4! = 24 possible orders in all, and we’ve only gone through 8 of them, so it’s too early to repeat. In fact, what this shows is that no matter how many times you apply these two transformations (and, note that both of them square to the identity, so there’s no use using higher powers) you’ll only go through these 8 permutations. This means that the two transformations we’re using generate only a subgroup of the full group of 24 permutations.

The trick to generate the rest of the permutations is to toss in an extra swap that’s not in this group, namely the exchange (CD). Then we can go ahead and apply the original pattern of alternating (AB)(CD) with (BC) again. This means that the next 8 extents are

-> 1342 -> 3124 -> 3214 -> 2341 -> 2431 -> 4213 -> 4123 -> 1432 ...

Again, because ((BC) (AB) (CD))^4 is the identity, we don’t apply that last (BC) swap. Instead, we apply another (CD) exchange as before:

-> 1423 -> 4132 -> 4312 -> 3421 -> 3241 -> 2314 -> 2134 -> 1243 ...

and then a final (CD) swap brings us home to the original extent 1234.
A mathematical way of understanding the structure of Plain Bob Minimus is to think in terms of cosets of the subgroup. In general, suppose H is a subgroup of a group G. When we multiply everything in H by an element k not in H, we get another subset H k of the group that has no overlap with H. (When two subsets have no overlap, we say they are disjoint.) In fact, by multiplying H by various elements not in H, we can dissect the whole group into disjoint cosets:

\[ G = H \cup H k \cup H k^2 \cup \ldots \]

In the case at hand, G is the whole group of permutations on 4 positions, which has size 4!=24, and H is the subgroup generated by (AB)(CD) and (BC), which has size 8. So, we can slice G into three disjoint cosets of H:

\[ G = H \cup H k \cup H k^2, \quad k = (CD)(BC)=(CBD) \]

So, the first 8 extents arise by applying permutations in H (including the identity) to the initial 1234; the next 8 come from applying permutations in the coset H k to 1234; and the last come from applying permutations in H k^2 to 1234.

Where does k = (CD)(BC) H come from? When we have finished the first 8 extents, one more (BC) operation would bring us back to 1234. Instead, we apply (CD). So, the first extent 1342 of the middle set can be thought of as obtained by applying (CD)(BC)^{-1} to 1234. And, since (BC) is its own inverse, this is the same as applying k=(CD)(BC) to 1234. Everything else in the middle set of 8 extends is obtained by applying permutations in the coset Hk to 1234. Similarly, the last 8 extents come from applying permutations in coset Hk^2 to 1234.

Note: In a non-Abelian group, there are two different ways of forming cosets of a subgroup H: you can multiply on the left, generating left cosets k H, or you can multiply on the right, generating right cosets of the form H k. We can either slice G into right cosets of H, or into left cosets of H, but in general these two ways of slicing are not the same.
But, the definition of a normal subgroup is that every right coset is also a left coset; in fact, if $H$ is normal then for any $k$ in $G$ we have $Hk = kH$.

Assignment: Read Harkelroad Chapter 5.

Problems:
1. Decide if the following permutations of 5 bells are even or odd:
   (a) $12345 \rightarrow 13425$  
   (b) $12345 \rightarrow 54312$
   (See Harkelroad p. 68-69.)

2. Problem on “Plain Hunt”, Benson page 330.

3. Suppose you want to ring the changes on 4 bells. Show that the repeating the sequence of transformations $(AB)$ then $(BC)$ then $(CD)$ will not generate all the permutations, and suggest how to modify this to sequence to get them all. (Remember that we can’t return to the starting configuration 1234 until the bells have been rung.)