# Lecture 7:
# Plucked Strings and the Wave Equation

Here we want to look in more detail at how the string on a guitar (or violin) vibrates when plucked. Specifically, we'll look at how different points along the string move transverse to the length of the string. Our goal will be to explain the harmonics of the note produced by the string—i.e., why their frequencies are all integer multiples of the lowest frequency heard (the fundamental).

---

Let $x$ be a horizontal coordinate measuring position along a string of length $\ell$ when it's at rest, and let $y(x,t)$ be the transverse displacement of the string at position $x$ and time $t$. Say that the string has tension $T$ along it (see Figure 1). In guitar and violin strings, this tension is produced by fixing one end of the string at the bottom of the instrument, threading the other end of the string through a tuning peg at the top of the instrument, and tightening the peg.

We want to find a differential equation satisfied by $y(x,t)$, using Newton's law $F = ma$. We focus on a short segment of the string, between position $x$ and $x + \Delta x$. Let $F$ be the vertical force on the piece. At the ends, the tension force $T$ is directed along the string, whose slope is $\partial y/\partial x$ (see Benson Figure 3.2). If we let $\theta$ be the angle of elevation of the tangent to the string (so that $\tan\theta = \partial y/\partial x$), then the vertical component of the tension at any point is $T\sin\theta$ (see Figure 2). Thus, the net force on this segment of the string is

$$F = T\sin\theta(x+\Delta x)) - T\sin\theta(x).$$

(For the moment, we're suppressing dependence on $t$.)

<u>Small Displacement Approximation.</u> When the displacement $y$ is small, the angle $\theta$ is small as well, so that $\cos\theta \approx 1$ and $\sin\theta \approx \tan\theta$. So our

approximate sum of forces is

$$F = T\left(\frac{\partial y}{\partial x}(x+\Delta x) - \frac{\partial y}{\partial x}(x)\right).$$

Suppose the string has linear density $\rho$ (i.e., its mass is $\rho$ grams per millimetre). Then the mass of this piece of string is $m = \rho\Delta x$, the vertical acceleration is $a = \partial^2 y/\partial t^2$, so $F = ma$ gives

$$T\left(\frac{\partial y}{\partial x}(x+\Delta x) - \frac{\partial y}{\partial x}(x)\right) = \rho\Delta x \frac{\partial^2 y}{\partial t^2}(x).$$

Dividing by $\Delta x$ and letting $\Delta x \to 0$ gives us

$$T\frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}.$$

We recognize this as a form of the *wave equation*, with constants $T$ and $\rho$ thrown in. Usually these constants are lumped together into one constant, by letting $c = \sqrt{T/\rho}$, whereupon our equation becomes

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

We will see in moment that $c$ is the speed with which waves travel along the string.

For most PDE, there is no way of writing down a general solution; in fact, you're extremely lucky if you can find one non-trivial explicit solution. However, in the 18th century Jean d'Alembert figured out how to write down *all* the solutions of the wave equation. This result comes from writing the equation in the following way,

$$\left(\left(\frac{\partial}{\partial t}\right)^2 - c^2\left(\frac{\partial}{\partial x}\right)^2\right) y = 0,$$

and factoring the operator on the left-hand side as a composition:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) y = 0.$$

(Note that the middle terms in the product cancel out.) The idea is that if we had a function $y(x,t)$ such that $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) y = 0$, then it would solve the wave equation. Examples of such functions are

$$y = x + ct, \text{ or more generally } y = f(x + ct).$$

What do these solutions look like? By comparing the graphs at time zero and at a later time (see Figure 3), we see that the graph just travels to the left by $c$ units per unit time. In other words, such solutions are 'waves' that travel to the left (while maintaining their shape) with speed $c$.

We can also factor the wave equation as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) y = 0,$$

and this shows that solutions of $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) y = 0$ are also solutions of the wave equation. Such functions take the form $y = g(x - ct)$, and are *rightward* travelling waves. Because the wave equation is linear and homogeneous, we can add these two kinds of solutions together, giving the *d'Alembert solution*

$$y(x,t) = f(x + ct) + g(x - ct).$$

This is the *general solution* of the wave equation; in other words, all solutions can be written as a sum of a leftward travelling wave and a rightward travelling wave.

---

Now, the wave on our string can't travel forever, because the ends of the string are fixed:

$$y(0,t) = 0 = y(\ell, t) \quad \text{for all } t.$$

These are the *boundary conditions* that we want the solutions of our PDE to satisfy.

Substituting the d'Alembert solution into the boundary condition at $x = 0$ gives

$$0 = f(ct) + g(-ct) \quad \text{for all } t,$$

hence $g(u) = -f(u)$ for any real number $u$. We can then write

$$y(x,t) = f(x + ct) - f(-x + ct). \tag{1}$$

(Note that this implies that, for any $t$, $y$ and $\partial y/\partial t$ are *odd* functions of $x$.) Imposing the boundary condition at $x = \ell$ gives $0 = f(\ell+ct) - f(-\ell+ct)$, which in turn implies that

$$f(u + \ell) = f(u - \ell) \tag{2}$$

for any real number $u$. This is equivalent to $f$ being periodic with period $2\ell$, since $f(u+2\ell) = f(u+\ell+\ell)$ and then using $(2)$ shows that $f(u+2\ell) = f(u+\ell-\ell) = f(u)$.

Our study of Fourier series shows that (most) functions of period $2\ell$ can be represented as a sum

$$f(x) = \sum_{n=1}^{\infty} c_n \cos\left(n\frac{\pi x}{\ell} + \phi_k\right).$$

(Note that we've left off the constant term in the Fourier series, since any constant added to $f$ would cancel out in the formula $(1)$ for $y(x,t)$.) Each of the terms in this series gives rise to a separate solution to the wave equation, satisfying the boundary conditions. Plugging the $n$th term into $(1)$ gives

$$y = c_n\left[\cos\left(n\frac{\pi(x+ct)}{\ell} + \phi_k\right) - \cos\left(n\frac{\pi(-x+ct)}{\ell} + \phi_k\right)\right].$$

Using $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$, with $A = \dfrac{n\pi ct}{\ell} + \phi$ and $B = \dfrac{n\pi x}{\ell}$, we get

$$y = -2c_n \sin\left(\frac{n\pi ct}{\ell} + \phi\right) \sin\left(\frac{n\pi x}{\ell}\right).$$

This tells us that the string has modes of vibration that look like a sine function with $n$ arches along the length of the string (in the $x$ direction), and which oscillate in time with frequency $(n\pi c/\ell)/(2\pi) = nc/(2\ell)$. In other words, the $n$th mode has time frequency exactly $n$ times the frequency of the lowest mode. This is the observation of the ancient Greeks (attributed to Pythagoras) that we first mentioned in Lecture 3.

Initial Conditions. The actual form of the function $f$ in the solution $(1)$ will be determined by the initial displacement and velocity of the string:

$$s_0(x) = y(x,0) = f(x) - f(-x), \tag{3}$$

$$v_0(x) = \frac{\partial y}{\partial t}(x,0) = c(f'(x) - f'(-x)). \tag{4}$$

Using these formulas we can solve for $f$ (and hence $y(x,t)$) by integration. Namely, integrating $(4)$ gives

$$\begin{aligned} \tfrac{1}{c}\int_0^x v_0(u)\,du &= \int_0^x f'(u) - f'(-u)\,du \\ &= f(x) + f(-x) + A \end{aligned}$$

where $A$ is some constant. (Note that Benson incorrectly leaves this constant out.) Adding this with $(3)$ and solving for $f(x)$ gives

$$f(x) = \tfrac{1}{2}\left(s_0(x) + \tfrac{1}{c}\int_0^x v_0(u)\,du - A\right).$$

Then substituting this in $(1)$ gives

$$\begin{aligned} y(x,t) &= \tfrac{1}{2}\left(s_0(x+ct) - s_0(-x+ct)\right) \\ &\quad + \tfrac{1}{2c}\left(\int_0^{x+ct} v_0(u)\,du + A - \int_0^{-x+ct} v_0(u)\,du - A\right) \\ &= \tfrac{1}{2}\left(s_0(x+ct) + s_0(x-ct)\right) + \tfrac{1}{2c}\int_{-x+ct}^{x+ct} v_0(u)\,du \end{aligned}$$

Note that the $A$ cancels out, and we've used the fact that $s_0$ is odd. Because $v_0(x)$ is odd, the integral on the right can be rewritten as

$$\int_{-x+ct}^{x+ct} v_0(u)\,du = \int_{-x+ct}^{x-ct} v_0(u)\,du + \int_{x-ct}^{x+ct} v_0(u)\,du = 0 + \int_{x-ct}^{x+ct} v_0(u)\,du,$$

so that we get the formula

$$(5) \qquad y(x,t) = \tfrac{1}{2}\left(s_0(x+ct) + s_0(x-ct)\right) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} v_0(u)\,du$$

(cf. Benson, page 101). This formula can be interpreted as saying, the value of $y(x,t)$ is the average of the initial position of the string at the points $x+ct$ and $x-ct$, plus the average value of the initial velocity between those two points. In particular, if you draw lines from the point $(x,t)$ backwards in time, with slope $1/c$, and shade in the triangle formed by the intersections of these lines with the line $t=0$, then the value of $y(x,t)$ depends only on initial data along the part of the line $t=0$ that forms the base of this triangle (see Figure 4).

Plucking. For the moment, suppose that we have the special situation where $v_0(x)$ is identically zero and $s_0(x)$ looks like two sides of a triangle, where the remaining side is a line segment of length $\ell$ along the $x$-axis. Because $s_0(x) = y(x,0)$ is odd and $2\ell$-periodic, we extend it to all $x$ by rotating the triangle by 180 degrees around one end of the segment, then horizontally translating that figure by multiples of $2\ell$ units to the left and right (see Figures 3.7 and 3.8 in Benson).

Because $v_0(x) = 0$, $f'(x)$ is even, and $f(x)$ is odd, with $f(x) = \frac{1}{2}s_0(x)$. This means that the shape $y(x,t)$ of the plucked string at subsequent times is the sum of two copies of $\frac{1}{2}s_0(x)$, one travelling to the right and one travelling to the left.

*[Animate with Mathematica.]*

Assignment. Read Benson sections 3.1 through 3.3

Problems: Do p.99 #1, 2, and the following

A. Verify that the d'Alembert formula $y(x,t) = f(x+ct) + g(x-ct)$, where $f$ and $g$ are differentiable functions, satisfies the the wave equation.

B. Suppose that $y(x,t)$ is a solution of the wave equation equation with $v_0(x) = 0$ and

$$s_0(x) = \begin{cases} 1 \text{ when } -1 < x < 1, \\ 0 \text{ otherwise} \end{cases}$$

Sketch the graphs of $y(x, \frac{1}{2})$, $y(x,1)$ and $y(x,2)$. (Hint: Formula $(5)$ may be helpful.)

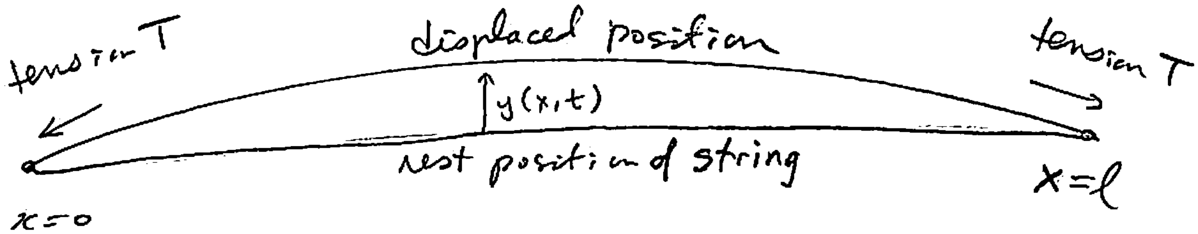


Figure 1

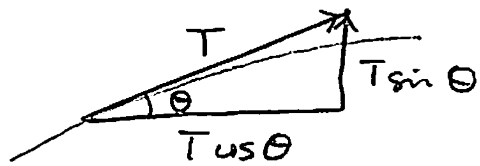


Figure 2

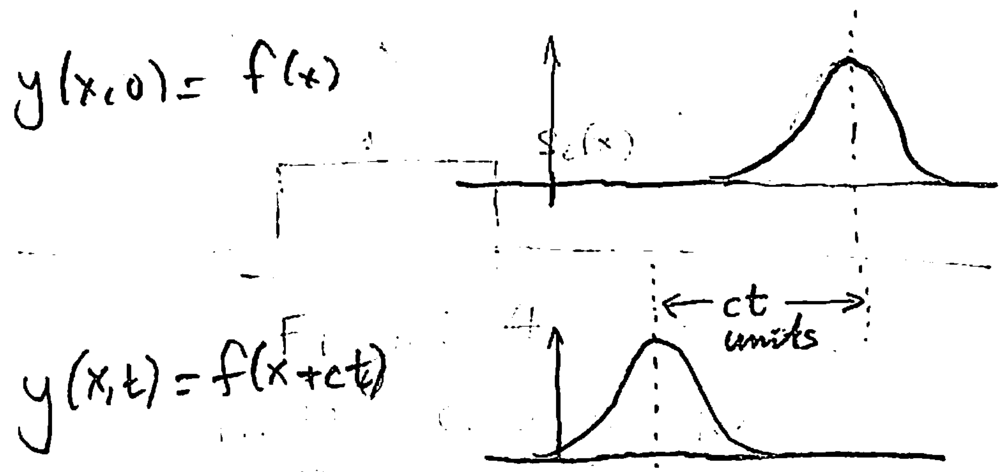


Figure 3 : leftward travelling wave

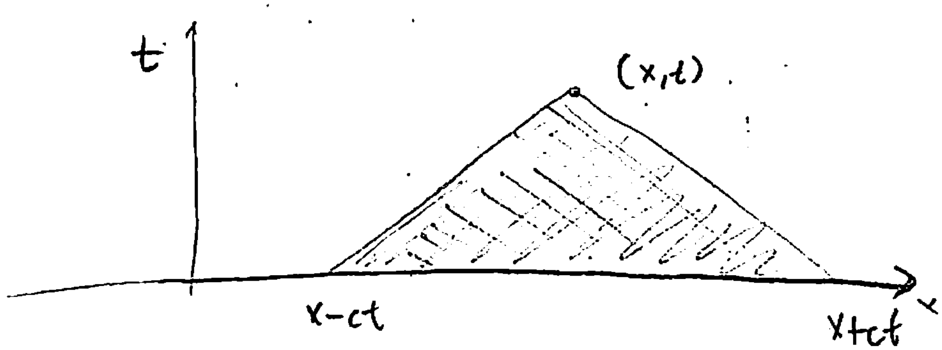


Figure 4