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ON SOLITONS FOR THE RICCI FLOW

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Abstract

In this thesis we investigate conditions for the existence of solitons for the Ricci flow. The Ricci flow, first introduced by Richard Hamilton, changes a Riemannian metric over time, in a way that the metric satisfies the partial differential equation $\partial g/\partial t = -2\,\mathrm{Ric}(g)$. "Solitons" for this flow are solutions of the equation where the metrics at different times differ by a diffeomorphism of the manifold. The soliton condition, sufficient for an initial metric to give rise to a soliton, is $\mathcal{L}_{\mathcal{X}}\} = -\in \mathrm{Ric}(\})$. We use the techniques of exterior differential systems to show that this condition is involutive, and gives an elliptic equation in harmonic coordinates. However, globally-defined solitons are harder to obtain on compact manifolds than locally-defined solitons: using techniques from Hamilton's earlier papers, we show that the only solitons on compact three-manifolds are metrics of constant curvature. For compact manifolds of higher dimension we investigate the possibility of deforming an Einstein metric to a soliton, and we calculate the first-order deformation space explicitly in the case of symmetric spaces of compact type. Using warped products and ODE techniques, we also construct some examples of non-compact, complete solitons.

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Contents

Introduction

What is the Ricci flow?

The Ricci flow is a way of evolving a Riemannian metric over time. The evolution is specified by the partial differential equation

$$\frac{\partial g}{\partial t} = -2 \ \mathrm{Rc}(g),$$

which is known as the "heat equation" for Riemannian metrics. There are situations — most notably, metrics with positive Ricci curvature on compact three-manifolds — where this name is justified, and the behaviour of the flow is indeed similar to that of the ordinary one-dimensional heat equation $f_t = f_{xx}$: if the initial metric g_0 has its curvature concentrated in a 'lump' in one place, the Ricci flow smoothes out the lump, so that as time approaches infinity the curvature is uniformly distributed. However, the nonlinear second-order operator Rc, which yields the Ricci tensor of the metric g, is not elliptic, so the Ricci flow is not strictly parabolic. The way in which it fails to be parabolic is understood well enough that it has been termed weakly parabolic (see [17], § 5,6).

What is it good for?

Ever since Riemannian geometry was introduced in the nineteenth century, curvature has been used to study the topology of manifolds. Of course, the classic result in this area is the Gauss-Bonnet theorem: if M is a compact orientable surface without boundary then the formula

$$\frac{1}{2\pi}\int_M K\,dA \;=\; \chi(M)$$

relates a curvature integral to the Euler characteristic $\chi(M)$, a topological invariant of M. Now, introducing a metric on a manifold adds an extra level of complication, since the space of all available metrics is extremely large. We can try to understand this complication by considering questions like

> Is there a "nicest" metric we can put on M, that is, one where the curvature is as simple as possible?
>
> How does the topology of M influence the curvature of metrics on M? In particular, how "nice" a metric can we get?

For example, if M is a two-dimensional torus the Gauss-Bonnet formula shows that M cannot have a metric of strictly negative or strictly positive curvature. On the other hand, the uniformization theorem says that an arbitrary complete metric on any surface (compact or not) can be multiplied by a smooth function to obtain a very nice metric, one of constant curvature.

Sometimes if a manifold admits a metric that is nice enough, we can completely determine its topology. For example, if M is a simply-connected surface admitting a complete metric of constant negative curvature, then M must be the upper half-plane. Even if M is not simply-connected, we can lift the constant negative curvature metric to its universal cover $\tilde{M}$, which then must be the upper half-plane. Furthermore, the deck transformations for this covering must be isometries of $\tilde{M}$, so they must lie in $SL(2, \mathbb{R})$. It follows that any hyperbolic surface M must be a quotient of the upper half-plane by the action of a subgroup of $SL(2, \mathbb{R})$. In this sense, the upper half-plane is a *model space* for hyperbolic surfaces.

In the next dimension up, Thurston [28] has conjectured that three-manifold topologies can be classified by a generalization of this argument. In brief, one hopes to show that every compact three-dimensional manifold can be sliced up into pieces, each of which admits a metric nice enough so that its universal cover is one of eight homogeneous model spaces. The goal, in essence, is to be able to use our understanding of nice geometries to investigate questions in topology.

The Ricci flow comes in as a way of obtaining nice geometry. In general, one starts with a metric g_0 on M that satisfies some rather general curvature condition $\mathcal{C}$, and proves that as the Ricci flow runs, the metrics g_t converge to a limiting metric which satisfies a more restrictive — "nicer" — condition $\mathcal{C}'$. For example, it is known that:

- If M is a compact three-manifold and g_0 has positive Ricci curvature, then g_t converges to a metric of constant positive sectional curvature [17]. Thus, M must be a quotient of S^3 by standard isometries.

- If M is a compact orientable surface, then g_t converges to the constant curvature metric appropriate to the topology of M — regardless of the initial metric g_0 [19],[10].

- If M is a compact four-manifold and the Riemann curvature tensor of g_0, when viewed as a quadratic form on the space of two-forms on M, is positive semi-definite, then g_t converges to a metric which is locally isometric to one of S^4, $\mathbb{CP}^{\nvDash}$, $S^3 \times \mathbb{R}$, $S^2 \times S^2$, or $S^2 \times \mathbb{R}^{\nvDash}$, with the standard metric in each case [18]. Thus M must be a quotient of one of these spaces.

- If M is a compact complex manifold and g_0 is a Kähler metric with non-negative holomorphic bisectional curvature, then g_t remains Kähler and has positive holomorphic bisectional curvature for large enough t. It follows by algebro-geometric arguments that M is biholomorphic to $\mathbb{CP}^{\nvDash}$ [25].

Strictly speaking, the above results for compact manifolds apply to the *normalized Ricci flow*

$$\frac{\partial g}{\partial t} = -2\ \mathrm{Rc}(g) + \frac{2}{n}\, r(g)\, g,$$

where $r(g)$ is the integral of the scalar curvature divided by the volume. This normalization is rigged so that, while the metric changes, the volume of the manifold is constant. (Without this normalization, S^n with constant curvature $+1$ shrinks to a point in time $1/2(n-1)$. On the other hand, for any Einstein metric the right-hand side of the above equation is zero, and the normalized Ricci flow leaves the metric fixed.) Since the two flows are equivalent up to re-scaling and reparametrizing in time (see Chapter 3), one usually works with the curvature evolution equations for the unnormalized flow and proves convergence results for the normalized flow.

However, given an arbitrary initial metric on a compact three-manifold, the normalized Ricci flow may develop singularities in finite time. For example, it is believed[1] that if we give the three-sphere a metric so that the lateral S^2's pinch in tightly in the middle, in finite time the Ricci flow will collapse one of the S^2's to a point. (See Figure 0.1 for the analogous picture in two dimensions; this phenomenon has been shown to occur for the mean curvature

Figure 0.1: A neck pinch

flow for embedded surfaces in $\mathbb{R}^3$ [16].)

The most ambitious conjecture for the Ricci flow is that the singularities that develop are due to the topology of M^3. Indeed, it is conjectured that the singularities split M^3 into less complicated pieces, and that away from the singularities the Ricci flow is converging to one of the homogeneous geometries proposed in the Thurston programme. To make progress along these lines, we must try to understand the singularities of the Ricci flow on three-manifolds.

What are Ricci solitons?

In this subject it has become customary to call a solution of an evolution equation that evolves along symmetries of the equation a *soliton*; another term

[1]This is claimed in the introduction to [22] but no references are given for the appropriate numerical studies.

used is *self-similar solution* (cf. [11]). To illustrate what we mean by "evolving along symmetries", consider the ordinary one-dimensional heat equation $f_t = f_{xx}$. It is not too hard to see that the following vector fields are infinitesimal symmetries of the equation — that is, each one generates a one-parameter group of transformations that transform solutions to solutions:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} && \text{translation in space} \\ X_2 &= \frac{\partial}{\partial t} && \text{translation in time} \\ X_3 &= f\frac{\partial}{\partial f} && \text{scaling in } f \\ X_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} && \text{scaling in space and time} \end{aligned}$$

(Note that by no means do these exhaust the symmetries of the heat equation; see [26], §2.4.) For the fundamental solution $f(x,t) = \dfrac{e^{-x^2/4t}}{\sqrt{t}}$, we see that

$$f(\lambda x, \lambda^2 t) = \lambda^{-2} f(x,t)$$

for any $\lambda > 0$, so this solution is a soliton that evolves along the vector field $X = X_4 - 2X_3$. (In fact, X is tangent to the graph of $f(x,t)$.)

Another well-known evolution equation is the curve-shortening flow

$$\frac{\partial X}{\partial t} = \kappa\, N,$$

for $X(t)$ a one-parameter family of immersed plane curves with curvature κ and unit normal vector N. The symmetries of this flow are the rigid motions of the plane, translation in time, and simultaneous dilation in space and time. Among the soliton solutions are the "spirograph" curves of Abresch-Langer [1], which evolve by rotation and dilation, and the "Grim Reaper" $\cos x = e^{-y}$, which evolves by translation in the y-direction.

Figure 0.2: (a) some Abresch-Langer curves (b) the Grim Reaper

The Ricci flow $\dfrac{\partial g}{\partial t} = -2\ \mathrm{Rc}(g)$ on a manifold M is an equation of tensors, so all the diffeomorphisms of M are symmetries of the Ricci flow. We define a *Ricci soliton* to be a solution where the metric g_t at time t is the pullback of the initial metric g_0 by a diffeomorphism ϕ_t. It follows (see Chapter 1) that the ϕ_t's belong to a 1-parameter group of diffeomorphisms generated by a vector field X. Then g_t is a Ricci soliton solution if and only if for some complete vector field X

$$\mathcal{L}_{\mathcal{X}}\,\}_{\prime} = -\in \mathrm{Rc}(\}_{\prime}).$$

We call this the Ricci soliton equation for an initial metric g_0; if g_0 satisfies this, then as the Ricci flow runs, g_t evolves along the vector field X. Notice that the metric g_t will be no nicer than our initial metric; in effect, the Ricci flow is changing the coordinates but not the geometry.

Why should we care about Ricci solitons?

If we imagine the Ricci flow as the flow of some vector field on the space of metrics, then solitons may act as attractors for this flow. In fact, in the case of two-dimensional spheres and spherical orbifolds, the flow is known to converge to a soliton. Given this tendency, it becomes important to classify the possible soliton metrics on a given compact manifold.

We are also interested in solitons on non-compact manifolds for the following reason. We can attempt to understand how singularities for the Ricci flow form by defining a sequence of re-scaled solutions. Suppose (p_k, T_k) is a sequence of points and times along which the norm of the curvature of $g(t)$ achieves its maximum at each time and is going to infinity as $k \to \infty$. For each k, re-scale so that the norm of the curvature is bounded at p_k:

$$g_k(t) = \lambda_k g(T_k + t/\lambda_k)$$

The re-scaling and translation in time makes each $g_k(t)$ a solution of the Ricci flow with bounded curvature at time zero. The boundedness of the product of the maximum norm of the curvature and the time until blow-up is invariant under these re-scalings. If it is bounded we say the singularity is "rapidly-forming"; if it is unbounded we say the singularity is "slowly-forming". In the latter case, if the metrics $g_k(0)$, which have uniformly bounded curvature as $k \to \infty$, converge to anything, that metric must be a solution to the Ricci flow for all time. The belief is that they converge to complete Ricci solitons. If this is true, then we can understand the singularities of the Ricci flow by understanding the solitons. (That a sequence of metrics on a compact manifold can converge to a non-compact soliton is not an unreasonable notion: it has been shown that, near a singularity, re-scaled solutions to the curve-shortening flow approach one of two soliton solutions, either the compact Abresch-Langer curves in the "rapidly-forming" case, or the Grim Reaper in the "slowly-forming" case [2].)

What is in this thesis?

In Chapter One, we review some standard material for the Ricci flow, particularly the curvature evolution equations and the notion of expanding and shrinking solitons.

In Chapter Two, we investigate the local solution space for the Ricci soliton equation using the techniques of exterior differential systems.

In Chapter Three, we prove that the only solitons on compact three-manifolds are metrics of constant sectional curvature.

In Chapter Four, we investigate the solution space for the Ricci soliton equation on compact manifolds by computing the deformation space of "trivial" compact solitons — i.e., Einstein metrics.

In Chapter Five, we construct new examples of non-compact, complete Ricci solitons in the form of a warped product of a Einstein manifold over $\mathbb{R}^{\ltimes}$ with a radially symmetric metric.

Chapter 1

Background for the Ricci Flow

Curvature Evolution and the Uhlenbeck trick

The unnormalized Ricci flow implies the following evolution equation for the curvature tensor:

$$\begin{aligned}\frac{\partial}{\partial t} R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &\quad - (R_i{}^p R_{pjkl} + R_j{}^p R_{ipkl} + R_k{}^p R_{ijpl} + R_l{}^p R_{ijkp}),\end{aligned}$$

where the Laplacian is simply the trace of iterated covariant derivatives and $B_{ijkl} = R_i{}^p{}_j{}^q R_{pkql}$. There is a trick, introduced in [18] and attributed there to Karen Uhlenbeck, that simplifies the quadratic terms on the right-hand side. Namely, assuming g obeys the Ricci flow, one constructs a gauge transformation $u : TM \to TM$ so that $h = u^* g$ does not vary with time.[1] In terms of a local frame for TM, suppose that $h_{ij} = u_i^k u_j^l g_{kl}$; then the components of u must obey

$$\frac{\partial}{\partial t} u_j^i = g^{ik} R_{kl} u_j^l.$$

[1] This idea is motivated nicely in the first part of [20].

If $\tilde{R}_{ijkl}$ denotes the pullback of the curvature tensor via u, then

$$\frac{\partial}{\partial t}\tilde{R}_{ijkl} = \Delta\tilde{R}_{ijkl} + 2(\tilde{B}_{ijkl} - \tilde{B}_{ijlk} + \tilde{B}_{ikjl} - \tilde{B}_{iljk}) \tag{1.1.1}$$

where the covariant derivatives in Δ are taken using the pullback of the Levi-Civita connection.

Expanding Solitons and the Harnack Inequality

In the Introduction we mentioned that the symmetries of the Ricci flow on M include the full group of diffeomorphisms of M. There is also a dilation symmetry: if $g_1(t)$ satisfies the Ricci flow, so does $g_2(t) = \lambda\, g_1(t/\lambda)$ for any nonzero constant λ. If $g_2 = g_1$ for all $\lambda > 0$ then the metric is an expanding solution for the Ricci flow:

$$g(t) = t\, g(1). \tag{1.1.2}$$

It follows that $g(t)$ shrinks to a point as $t \searrow 0$. If we plug (1.2) into the equation for the flow we get

$$g(1) = \frac{\partial}{\partial t} g(t) = -2\ \mathrm{Rc}(g(t)) = -2\ \mathrm{Rc}(g(1));$$

so, $g(1)$ is an Einstein metric with negative Ricci curvature.

We define an *expanding soliton* to be a solution of the Ricci flow where

$$g(t) = t\, \phi_t^* g(1),$$

for some family of diffeomorphisms ϕ_t, $t > 0$, with $\phi_1 = \mathrm{id}_M$. Note that we do not assume ϕ_t is a one-parameter group.

Proposition 1.1 *If $g(t)$ is an expanding soliton then there is a vector field X such that*

$$g(1) + \mathcal{L}_{\mathcal{X}}\}(\infty) = -\in \mathrm{Rc}(\}(\infty)). \tag{1.1.3}$$

If $g(1)$ has no Killing fields then X is uniquely determined, complete, and ϕ_t is the flow of X up to reparametrization. Conversely if (1.3) *holds for a complete X then $g(1)$ gives rise to an expanding soliton.*

On $M \times \mathbb{R}^+$ define the family of smooth maps

$$\Phi_s(p,t) = (\phi_{s+t} \circ \phi_t^{-1}(p), s+t)$$

for $s + t > 0$. In particular $\Phi_s(p,1) = \phi_{s+1}(p, s+1)$. For $s \geq 0$, Φ_s is a diffeomorphism and it is easy to check that $\Phi_{s_1} \circ \Phi_{s_2} = \Phi_{s_1+s_2}$. Let $\tilde{X}$ be the corresponding vector field on $M \times \mathbb{R}^+$. It is clear that $\tilde{X} = \frac{\partial}{\partial t} + \tilde{X}_H$, where $\tilde{X}_H$ is tangent to the fibres of the projection π_2 onto $\mathbb{R}^+$. Let $X_t = (\pi_1|_{M\times\{t\}})_* \tilde{X}$. If i is the inclusion of M as $M \times \{1\}$ in $M \times \mathbb{R}^+$, then $\phi_t = \pi_1 \circ \Phi_{t-1} \circ i$. For any covariant tensor ω on M

$$\begin{aligned} \frac{\partial}{\partial t}\phi_t^*\omega &= i^* \frac{\partial}{\partial t}\Phi_{t-1}^*(\pi_1^*\omega) \\ &= i^*\Phi_{t-1}^*(\mathcal{L}_{\tilde{\mathcal{X}}}\pi_\infty^*\omega) \\ &= \phi_t^*(\mathcal{L}_{\mathcal{X}_\sqcup}\omega) \end{aligned}$$

If $g(t)$ is to satisfy the Ricci flow, then

$$-2\ \mathrm{Rc}(g(t)) = \phi_t^* g(1) + t\frac{\partial}{\partial t}\phi_t^* g(1)$$

which implies

$$-2\ \mathrm{Rc}(g(1)) = g(1) + \mathcal{L}_{\sqcup\mathcal{X}_\sqcup}\}(\infty) \quad \forall \sqcup > \prime.$$

To get the first statement of the proposition, we can let $X = X_1$. If $g(1)$ has no Killing fields then $tX_t = X$ and the flow ψ_t of X is $\psi_t = \phi_{e^t}$, so X is complete. Similarly, to prove the converse we can let $\phi_t = \psi_{\log t}$ for ψ_t the flow of X.

Remark 1.2 *By similar calculations, we can show that* $g(t) = \phi_t^* g(1)$ *is equivalent to the Ricci soliton condition* $\mathcal{L}_\mathcal{X} \}(\prime) = -\in \mathrm{Rc}(\}(\prime))$ *(i.e. we may assume that* ϕ_t *is a 1-parameter group) and that a shrinking soliton*

$$g(t) = -tg(-1), \quad t < 0$$

is equivalent to $\mathcal{L}_\mathcal{X} \} + \} = +\in \mathrm{Rc}(\})$ *for* $g = g(-1)$*. Shrinking solitons only exist up finite time; for example, under the Ricci flow the standard metric on* S^n *shrinks to a point in finite time.*

Remark 1.3 *One reason that expanding solitons are interesting is that they constitute borderline cases for the Harnack inequality for the Ricci flow[20]. This asserts that when* $g(t)$ *follows the Ricci flow and has positive semi-definite curvature operator, a certain quadratic form on* $TM \oplus \Lambda^2 TM$ *is positive semi-definite. If* W^i *and* $U^{ij} = -U^{ji}$ *represent the components of a vector and bivector with respect to some local frame, then the assertion is that*

$$R_{ijkl}U_{ij}U^{kl} + 2(\nabla_i R_{jk} - \nabla_j R_{ik})U^{ij}W^k + M_{ij}W^iW^j \geq 0, \tag{1.1.4}$$

where

$$M_{ij} = \Delta R_{ij} - \frac{1}{2}\nabla_i\nabla_j R - R_{ik}R^k{}_j + 2R_{ikjl}R^{kl} + \frac{1}{2t}R_{ij}.$$

Now, if we have an expanding soliton, then at each time

$$\frac{1}{t}g + 2\,\mathrm{Rc}(g) = \mathcal{L}_{\mathcal{V}(\sqcup)}\}$$

for $V(t) = -\phi_{t*}^{-1} X(t)$. *Setting* $U = V \wedge W$ *makes the left-hand side of* (1.4) *zero for all* W. *Thus, expanding solitons show that the Harnack inequality obtained by Hamilton is optimal.*

Chapter 2

Local Analysis of the Soliton Equation

If we wish to classify Ricci solitons, it is helpful to start with the following questions:

> How large is the space of Ricci soliton metrics, modulo diffeomorphisms? For example, do they lie in a finite-dimensional space?
>
> Are there any integrability conditions implied by the soliton condition that may help us classify solitons?

In this chapter we will use the techniques of exterior differential systems to answer these questions. In particular, it will turn out that the soliton condition is involutive, and, modulo diffeomorphisms, the n-dimensional soliton metrics depend locally on $n^2 + n$ arbitrary functions of $n - 1$ variables. Prior to this analysis, it was not known how large this space was. We will also investigate the differential relations between a soliton metric g and the corresponding vector field that are implied by the Ricci soliton condition.

The meaning of involutivity

A system of k-th order PDE

$$F^\rho(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^i}, \dots, \frac{\partial^k u^\alpha}{\partial x^I}) = 0, \quad |I| = k, \quad \rho = 1 \dots r$$

for s unknown functions u^α of $x^1, \dots, x^n$ determines, and is in fact equivalent to, a submanifold $\mathcal{R}$ in the space $J^k(\mathbb{R}^\ltimes, \mathbb{R}^\sim)$ of k-jets of functions from $\mathbb{R}^\ltimes$ to $\mathbb{R}^\sim$. In fact if x^i, u^α, p^α_i, $p^\alpha_{ij}, \dots, p^\alpha_I$ are the usual coordinates on $J^k(\mathbb{R}^\ltimes, \mathbb{R}^\sim)$ then $\mathcal{R}$ is cut out by the functions

$$F^\rho(x^i, u^\alpha, p^\alpha_i, \dots, p^\alpha_I) = 0, \quad \rho = 1 \dots r$$

A solution of the PDE corresponds to an integral manifold $N \subset \mathcal{R}$ of the contact system on $J^k(\mathbb{R}^\ltimes, \mathbb{R}^\sim)$. We can also define the *prolongation* $\mathcal{R}^{(\parallel+\infty)} \subset \mathcal{J}^{\parallel+\infty}(\mathbb{R}^\ltimes, \mathbb{R}^\sim)$ of $\mathcal{R} = \mathcal{R}^{(\parallel)}$ as being cut out by the functions F^ρ and their total derivatives $\dfrac{DF^\rho}{Dx^i}$, $i = 1 \dots n$; similarly we can define $\mathcal{R}^{(\parallel+\in)}$, $\mathcal{R}^{(\parallel+\ni)}$, etc.

The "forgetful functors" $\pi_l : J^{l+1}(\mathbb{R}^\ltimes, \mathbb{R}^\sim) \to \mathbb{J}^{\lessdot}(\mathbb{R}^\ltimes, \mathbb{R}^\sim)$, which remember only the l-jet, are submersions, and of course $\pi_l(\mathcal{R}^{(\updownarrow+\infty)}) \subset \mathcal{R}^{(\updownarrow)}$. However, there is no guarantee that $\mathcal{R}^{(\updownarrow+\infty)}$ submerses onto $\mathcal{R}^{(\updownarrow)}$ via π_l. Put another way, there is no guarantee that an l-jet solution can be extended to an $(l+1)$-jet solution of the PDE.

To give a simple example, consider the following PDE for functions u and v of x and y:

$$u_x = v + f_1(x, y)$$

$$u_y = f_2(x, y)$$

These define a smooth manifold $\mathcal{R}^{(\infty)} \subset \mathcal{J}^\infty(\mathbb{R}^\nvDash, \mathbb{R}^\nvDash)$ of codimension two. $\mathcal{R}^{(\in)}$

is cut out by these equations and their derivatives

$$u_{xx} = v_x + \partial_x f_1$$

$$u_{xy} = v_y + \partial_y f_1$$

$$u_{xy} = \partial_x f_2$$

$$u_{yy} = \partial_y f_2$$

which imply that $v_y + \partial_y f_1 = \partial_x f_2$, i.e. $\mathcal{R}^{(\in)}$ lies over a submanifold of codimension *three* in $J^1(\mathbb{R}^{\nvDash}, \mathbb{R}^{\nvDash})$.

Another example is the PDE for isometrically embedding a Riemannian surface in Euclidean 3-space. Let g_{ij} be the metric with respect to local coordinates x^1, x^2 on the surface. The PDE for the map (u^1, u^2, u^3) into $\mathbb{R}^{\nVdash}$ is

$$\sum_\alpha \partial_i u^\alpha \partial_j u^\alpha = g_{ij} \qquad \alpha = 1, \ldots, 3,\ i, j = 1, 2$$

Differentiating by x^k gives, after a little manipulation,

$$\sum_\alpha \partial_i u^\alpha \partial_{jk} u^\alpha = \tfrac{1}{2}(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}) \tag{2.2.1}$$

and differentiating again gives

$$\sum_\alpha (\partial_{il} u^\alpha \partial_{jk} u^\alpha + \partial_i u^\alpha \partial_{jkl} u^\alpha) = \tfrac{1}{2}\partial_l(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}).$$

Let $[i, jk]$ stand for the left-hand side of (2.1). Then skew-symmetrizing in k and l in the last equation gives the Gauss equation

$$\sum_\alpha (\partial_{11} u^\alpha \partial_{22} u^\alpha - (\partial_{12} u^\alpha)^2) - g^{ij}([i, 11][j, 22] - [i, 12][j, 12]) = K(g_{11} g_{22} - g_{12}^2),$$

an extra condition on the 2-jets of the u^α that we obtained by differentiating twice and equating mixed partials. In this case $\mathcal{R}^{(\in)}$ submerses onto $\mathcal{R}^{(\infty)}$ but $\mathcal{R}^{(\ni)}$ does not submerse onto $\mathcal{R}^{(\in)}$.

When a system of PDE becomes involutive at the k-jet level, each $\mathcal{R}^{(\updownarrow+\infty)}$ submerses onto $\mathcal{R}^{(\updownarrow)}$ for $l \geq k$, and the rank of the submersion can be calculated (see [7], Thm IV.4.4). In fact, involution implies that not only is there no obstruction to extending a kth degree Taylor polynomial to a formal power series solution, but (in the case when the functions F^ρ are analytic) the power series will converge on a small neighbourhood in $\mathbb{R}^{\ltimes}$.

Ellipticity of the soliton condition

In light of Proposition 1.1, we will consider a slightly generalized soliton condition

$$\mathcal{L}_{\mathcal{X}}\} = \in \mathrm{Rc}(\}) + \lambda \} \tag{2.2.2}$$

where λ is some constant. In this section, we will show this is an elliptic equation.

As it stands the condition is underdetermined; and, since it is invariant under diffeomorphisms, the solution space is at least as big as n arbitrary functions of n variables. To get a determined equation, and mod out by the diffeomorphism group, we adjoin the condition that the local coordinates x^i be harmonic functions with respect to the metric[1]. This is actually a trace condition on the Christoffel symbols: $g^{ij}\Gamma^k_{ij} = 0$. Since any new harmonic coordinate is a solution of an elliptic equation, the space of diffeomorphisms that preserve this condition is finite-dimensional. Putting this together with (2.2) we get a system that is, by a naive count, determined, first order in X,

[1] This fairly well-known trick is originally due to DeTurck; see [5], Chapter 5.

and of mixed second-order and first-order in g:

$$\begin{cases} g^{ij}\Gamma^k_{ij} = 0 \\ R_{ij}(g) + \lambda\, g_{ij} - \nabla_i X_j - \nabla_j X_i = 0 \end{cases} \tag{2.2.3}$$

(Here X_i denotes components of the metric dual of X.) Although the principal symbol of this system is degenerate, it turns out to be elliptic in the generalized sense of Douglis-Nirenberg [12]. This implies that g and X will be analytic in these coordinates. Since any other harmonic coordinates y^i satisfy an elliptic equation, they will also be analytic functions of the coordinates. We conclude

Theorem 2.1 *If M has a metric g satisfying* (2.2)*, then M has the structure of a real analytic manifold with respect to which g and X are analytic.*

Rather than calculating the Douglis-Nirenberg condition, we will calculate the characteristic variety of the corresponding differential system. In fact, this is essentially the same calculation, and it will allow us to set up the machinery for the involutivity calculation.

The soliton system and its characteristic variety

Let $x^1 \ldots x^n$ be local coordinates on an open set $U \subset \mathbb{R}^\ltimes$, and let ω^i be shorthand for $d(x^i)$. Let $\Omega = \omega^1 \wedge \ldots \wedge \omega^n$, and let $\omega_{(i_i \ldots i_k)}$ be the wedge product of the ω's such that

$$\omega_{(i_i \ldots i_k)} \wedge \omega^{i_1} \wedge \ldots \wedge \omega^{i_k} = \Omega;$$

for example, $\omega_{(2)} = \omega^3 \wedge \omega^1$ when $n = 3$ and $\omega_{(41)} = -\omega^2 \wedge \omega^3$ when $n = 4$. It is easy to verify that

$$\omega_{(j)} \wedge \omega^i = \delta^i_j \Omega$$

$$\omega_{(jk)} \wedge \omega^i = \delta^i_j \omega_{(k)} - \delta^i_k \omega_{(j)}$$

$$\omega_{(ijk)} \wedge \omega^p = \delta^p_i \omega_{(jk)} - \delta^p_j \omega_{(ik)} + \delta^p_k \omega_{(ij)}$$

We will now set up our exterior differential system. Let U be an open set in $\mathbb{R}^{\ltimes}$, let S_+ denote the open set of positive definite symmetric $n \times n$ matrices g_{ij} and let C be a $n\binom{n+1}{2}$-dimensional vector space with coordinates $\Gamma^i_{jk} = \Gamma^i_{kj}$. Let g^{ij} denote the inverse of g_{ij}, and let $R \subset C$ be the codimension-n subspace cut out by $g^{ij}\Gamma^k_{ij} = 0$. Let f_i and f_{ij} be coordinates on $\mathbb{R}^{\ltimes}$ and $\mathbb{R}^{\ltimes^{\nvDash}}$. On $U \times S_+ \times R$ define the forms

$$\phi^i_j = \Gamma^i_{jk} \omega^k$$

$$\gamma^{ij} = dg^{ij} + g^{ik}\phi^j_k + g^{kj}\phi^i_k$$

$$\Phi^i_j = d\phi^i_j + \phi^i_k \wedge \phi^k_j$$

$$\Phi^{ij} = \Phi^i_k g^{kj}$$

A metric on U can be thought of as a section of $U \times S_+$, equivalently a n-dimensional submanifold which submerses onto U, equivalently one on which the n-form Ω does not vanish. If we have a section of $U \times S_+ \times C$ on which the forms γ^{ij} vanish, then the Γ^i_{jk} are the Christoffel symbols of the metric, $\Phi^i_j = \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l$, and one can show that the Ricci tensor appears naturally in the $(n-1)$-form

$$\Phi^{ij} \wedge \omega_{(pij)} = R^j_j \omega_{(p)} - 2R^j_p \omega_{(j)}.$$

If we want the f_i to be the components of the dual of X, we should also require that the following forms vanish on a section of $N = U \times S_+ \times R \times \mathbb{R}^{\ltimes} \times \mathbb{R}^{\ltimes^{\nvDash}}$, the space of 1-jets of solutions to (2.3):

$$\sigma_i = df_i - f_j\phi_i^j - f_{ij}\omega^j$$

$$\Theta_p = \Phi^{ij} \wedge \omega_{(pij)} + 2(f_{pj} + f_{jp})g^{jk}\omega_{(k)} - 2g^{ij}f_{ij}\omega_{(p)}$$

Note that the $(n-1)$-forms Θ_i encode the soliton condition. To the forms $\gamma^{ij}, \sigma_i, \Theta_i$ we need to add their differentials. If we define

$$\tau_{ij} = df_{ij} - f_{ik}\phi_j^k - f_{kj}\phi_i^k$$

$$\Psi^{ij} = \Phi^{ij} + \Phi^{ji}$$

then

$$d\gamma^{ij} \equiv \Psi^{ij} \quad \mathrm{mod}\{\gamma^{ij}\}$$

$$d\sigma_i \equiv -\tau_{ij} \wedge w^j - f_p\Phi_i^p \quad \mathrm{mod}\{\gamma^{ij}, \sigma_i\}$$

$$d\Theta_i \equiv 2g^{jk}(\tau_{ij}w_{(k)} + \tau_{ji}w_{(k)} - \tau_{jk}\omega_{(i)}) \quad \mathrm{mod}\{\gamma^{ij}, \sigma_i, \Psi^{ij}\}.$$

(In fact, if the vanishing of Θ_i corresponds to prescribing the Ricci tensor, then the vanishing of $d\Theta_i$ corresponds to the second Bianchi identity $g^{jk}(\nabla_i R_{jk} - 2\nabla_k R_{ij}) = 0$.)

At a point $x \in N$, an integral n-plane of this system is an n-dimensional subspace of T_xN to which the forms

$$\gamma^{ij}, \sigma_i, \Psi^{ij}, d\sigma_i, \Theta_i, d\Theta_i \tag{2.2.4}$$

restrict to be zero. We will only consider those integral n-planes E which satisfy the independence condition $\Omega|_E \neq 0$. Even so, at every point of N there are integral n-planes. This is because such an integral element is in fact a 2-jet of a metric and a 2-jet of a vector field satisfying (2.3). We can always find a metric on a small neighbourhood with prescribed curvature at one point by using power series. We can then obtain harmonic coordinates with the same 1-jet as our given coordinates in a neighbourhood of that point.

Given an integral n-plane $E \subset T_xN$ we can obtain a coframe ω^i, γ^{ij}, π^i_{jk}, σ_i, σ_{ij} for the larger space $U \times S_+ \times C \times \mathbb{R}^{\ltimes} \times \mathbb{R}^{\ltimes\nvDash}$ at x, with $\pi^i_{jk} = \pi^i_{kj}$, $\pi^i_{jk} \equiv d\Gamma^i_{jk} \mod \omega^i$, $\sigma_{ij} \equiv \tau_{ij} \mod \omega^i$, such that

$$\begin{cases} g^{jk}\pi^i_{jk}|_{T_xN} = 0 \\ \Psi^{ij} = (g^{jk}\pi^i_{kl} + g^{ik}\pi^j_{il}) \wedge \omega^l \\ d\sigma_i \equiv \sigma_{ij} \wedge \omega^j + f_p\pi^p_{il} \wedge \omega^l \\ \Theta_i = g^{kp}\pi^j_{pl} \wedge \omega^l \wedge \omega_{(ijk)} \\ d\Theta_i \equiv 2g^{jk}(\sigma_{ij}w_{(k)} + \sigma_{ji}w_{(k)} - \sigma_{jk}\omega_{(i)}) \end{cases} \tag{2.2.5}$$

and π^i_{jk}, σ_{ij} vanish on E. We will refer to this as a torsion-absorbed coframe. On any other integral n-plane at x the 1-forms of (2.4) vanish and $\pi^i_{jk} = s^i_{jkl}\omega^l$, $\sigma_{ij} = t_{ijk}\omega^k$, where the s^i_{jkl} and t_{ijk} must satisfy some homogeneous linear equations dictated by (2.5). Thus, the space of integral n-planes naturally forms an affine bundle over N; we will calculate the rank of this bundle later.

A hyperplane in E is characteristic if it is contained in more than one integral n-plane. The characteristic variety Ξ_E is the set of directions in $\mathbb{P}(\mathbb{E}^*)$ that cut out characteristic hyperplanes, and the complex characteristic variety $\Xi^{\mathbb{C}}_E$ is cut out by the same equations considered over $\mathbb{C}$. There is a simple-minded algorithm for calculating the characteristic variety by duality.

Proposition 2.2 *Let E be an integral n-plane of an arbitrary exterior differential system and let ω^i, π^a be a coframe with $\pi^a|_E = 0$. Given a nonzero $\xi \in E^*$ (expressed as a linear combination of the ω's), for each form ψ in the system of degree at most n there is an expansion*

$$\psi \wedge \xi = \sum_I \pi^\psi_I \wedge \omega^I + \dots$$

where for each multi-index I, π_I^ψ is some linear combination of the π^a and we leave off wedge products of two or more π^a's. (If the system is linear there will be none of these anyway.) Then ξ is in the (deprojectivized) characteristic variety exactly when the span of all the 1-forms π_I^ψ, taken over all algebraic generators ψ of the system of degree at most n, is a proper subspace of the span of the π^a.

The hyperplane $\xi^\perp$ is characteristic when its polar space

$$H(\xi^\perp) = \{v \in T_x N | (v \lrcorner \psi)|_{\xi^\perp} = 0, \forall \text{ generators } \psi\}$$

is larger than E (cf. Defn. III.1.5 and V.1.1 in [7]). The condition on v is equivalent to

$$((v \lrcorner \psi) \wedge \xi)|_E = 0;$$

since ψ vanishes on E this is equivalent to

$$(v \lrcorner (\psi \wedge \xi))|_E = 0.$$

Using the above expansion for $\psi \wedge \xi$ yields

$$H(\xi^\perp) = \{v \in T_x N | v \lrcorner \pi_I^\psi = 0, \forall \psi, I\}.$$

Proposition 2.3 *If $\xi \in E^* \otimes \mathbb{C}$ is characteristic for E an integral n-plane of (2.4), then $g^{ij}\xi_i\xi_j = 0$. Since g_{ij} is positive definite this implies there are no real characteristics.*

Let $\xi^i = g^{ij}\xi_k$, $|\xi|^2 = g^{km}\xi_k\xi_m$, and

$$\rho_{ijk} = g_{ip}\pi^p_{jk} + g_{jp}\pi^p_{ik}.$$

(Note that $g^{jk}\pi^i_{jk} = 0$ implies $g^{jk}\rho_{jki} = 2g^{jk}\rho_{ijk}$.) The forms that cut out $H(\xi^\perp)$ comprise γ^{ij}, σ_i, and the 1-forms

$$\begin{aligned}\psi_{ijlm} &= \xi_l\rho_{ijm} - \xi_m\rho_{ijl}\\ \sigma_{ijk} &= \xi_k(\sigma_{ij} + f_l\pi^l_{ij}) - \xi_j(\sigma_{ik} + f_l\pi^l_{ik})\\ \theta_{ij} &= \xi^k(\rho_{kij} - \rho_{ijk} + \pi^p_{kp}g_{ij}) - \xi_i\pi^k_{jk}\\ \alpha_i &= (\sigma_{ij} + \sigma_{ji})\xi^j - g^{jk}\sigma_{jk}\xi_i\end{aligned}$$

which arise from wedging ξ with Ψ^{ij}, $d\sigma_i$, Θ_i, and $d\Theta_i$ respectively. One can calculate that if

$$\beta_{ij} = \frac{1}{2}(\theta_{ij} + \theta_{ji} + g^{km}(\psi_{ikjm} + \psi_{jkim})) = -\rho_{ijk}\xi^k + g_{ij}g^{lm}\rho_{lmk}\xi^k$$

then

$$\psi_{ijlm}\xi^m + \xi_l(\beta_{ij} - \frac{1}{n-2}g_{ij}g^{kl}\beta_{kl}) = -|\xi|^2\rho_{ijl};$$

also,

$$\xi^k(\sigma_{ijk} - \sigma_{jik} - \sigma_{kij}) + \xi_j\alpha_i - \xi_i\alpha_j \equiv |\xi|^2(\sigma_{ij} - \sigma_{ji}) \quad \mathrm{mod}\{\pi^i_{jk}\}$$

and

$$\xi^k(\sigma_{ijk}+\sigma_{jik})+\frac{1}{2}(\xi_i\alpha_j+\xi_j\alpha_i) \equiv |\xi|^2(\sigma_{ij}+\sigma_{ji})-g^{lm}\sigma_{lm}\xi_i\xi_j \quad \mathrm{mod}\{\pi^i_{jk}, \sigma_{ij}-\sigma_{ji}\}.$$

This shows that if $|\xi|^2 = g^{ij}\xi_i\xi_j \neq 0$, the span of the 1-forms $\psi_{ijlm}, \sigma_{ijk}, \theta_{ij}, \alpha_i$ is the same as that of π^i_{jk}, σ_{ij}, and $H(\xi^\perp) = E$.

Testing for Involutivity

First we will calculate the rank of the bundle of integral n-planes over each point of N.

Proposition 2.4 *At each point of N the space of integral n-planes of* (2.4) *has dimension*

$$\binom{n+1}{2}^2 - n^2 - \binom{n+1}{2} + n\binom{n+1}{2} - n.$$

To obtain an integral n-plane, set $\pi^i_{jk} = g^{ip}s_{pjkl}\omega^l$, $\sigma_{ij} = t_{ijk}\omega^k$, where the t_{ijk} have no symmetries but $s_{ijkl} = s_{ikjl}$, which we abbreviate by saying $s_{ijkl} \in V \otimes S^2V \otimes V$. When we substitute into (2.5), the resulting linear equations for the s_{ijkl} and t_{ijk} are

$$g^{jk}s_{ijkl} = 0 \tag{2.2.6}$$

$$s_{ijkl} + s_{jikl} - s_{ijlk} - s_{jilk} = 0 \tag{2.2.7}$$

$$t_{ikl} - t_{ilk} = -f_p g^{pj}(s_{jikl} - s_{jilk}) \tag{2.2.8}$$

$$g^{jk}(s^l_{jli} - s^l_{jil}) - g^{jl}(s^k_{jli} - s^k_{jil}) + g^{lm}(s^j_{lmj} - s^j_{ljm})\delta^k_i = 0 \tag{2.2.9}$$

$$g^{jk}(t_{ijk} + t_{jik} - t_{jki}) = 0 \tag{2.2.10}$$

where $s^i_{jkl} = g^{ip}s_{pjkl}$. The linear map in (2.7) can be factored as

$$V \otimes S^2V \otimes V \xrightarrow{\cong} S^2V \otimes V \otimes V \overset{\wedge_{34}}{\twoheadrightarrow} S^2V \otimes \Lambda^2 V$$

where the first map is symmetrization on factors 1 and 2. The kernel of the second map is clearly $S^2V \otimes S^2V$; so, to solve (2.7) set

$$s_{ijkl} = b_{ijkl} + b_{ikjl} - b_{jkil}$$

for $b_{ijkl} \in S^2V \otimes S^2V$. Then (2.6) and (2.9) become

$$(2b_{ijkl} - b_{jkil})g^{jk} = 0 \quad \forall i, l \tag{2.2.11}$$

$$g^{jl}b_{ikjl} - g_{ik}g^{jl}g^{mp}b_{jmlp} = 0 \quad \forall i, k \tag{2.2.12}$$

For any $a_{ij} \in S^2_o V$ (i.e. such that $g^{ij}a_{ij} = 0$), $b_{ijkl} = (n-2)a_{ij}g_{kl} + 2g_{ij}a_{kl}$ is a solution of (2.11). Substituting this in the left-hand side of (2.12) yields $n(n-2)a_{ij}$. Meanwhile $b_{ijkl} = 2ng_{ij}g_{kl} + (n-2)(g_{il}g_{jk} + g_{jl}g_{ik})$ also satisfies (2.11) and substituting this into (2.12) yields $(n^2+n-2)(2-n)g_{ik}$. Together, these two calculations show that the kernel of the linear map in (2.11) surjects onto S^2V under the linear map in (2.12). Thus the dimension of the space of s_{ijkl} satisfying (2.6), (2.7), (2.9) is $\binom{n+1}{2}^2 - n^2 - \binom{n+1}{2}$.

On the other hand the linear map on the t_{ijk} in (2.8) is clearly surjective with kernel $V \otimes S^2V$ and it is easy to show that this kernel surjects onto V under the linear map in (2.10). Thus given a solution s_{ijkl} of (2.6), (2.7), (2.9), the dimension of the solution space of the remaining equations for the t_{ijk} is $n\binom{n+1}{2} - n$.

Using the algorithm contained in Prop. III.1.15 in [7], we will calculate the Cartan characters $s_1, \cdots s_n$ of (2.4) with respect to the flag

$$0 \subset E_1 \subset \cdots E_{n-1} \subset E_n = E,$$

where $E_j \subset E$ is annihilated by the forms ω^k, $k > j$. If we use $\rho_{ijk} = g_{ip}\pi^p_{jk} + g_{jp}\pi^p_{ik}$ as before, the structure equations (2.5) become

$$2g^{jk}\rho_{ijk} = g^{jk}\rho_{jki} \tag{2.2.13}$$

$$\Psi_{ij} = \rho_{ijk}\omega^k \tag{2.2.14}$$

$$d\sigma_i \equiv \sigma_{ij} \wedge \omega^j \quad \mod\{\rho_{ijk}\} \tag{2.2.15}$$

$$(-1)^{n-3}\, 2\Theta_i = g^{jk}g^{lm}\rho_{ikl}\omega_{(jm)} - g^{jk}g^{lm}\rho_{lmk}\omega_{(ij)} \tag{2.2.16}$$

$$d\Theta_i \equiv 2g^{jk}(\sigma_{ij}w_{(k)} + \sigma_{ji}w_{(k)} - \sigma_{jk}\omega_{(i)}). \tag{2.2.17}$$

When $n > 3$, we can inspect the 2-forms in (2.14), (2.15) to calculate the characters $s_1, \cdots, s_{n-3}$. For $m < n-2$ we find

$$s_1 + \cdots + s_m = m\binom{n+1}{2} + mn$$

where the first term comes from the ρ_{ijk}, $k \le m$, and the second from the σ_{ik}, $k \le m$. By the same count $s_{n-2} \ge \binom{n+1}{2} + n$, but we must also take into account the extra polar space annihilators that occur as coefficients of $\omega_{(n-1,n)}$ in (2.16).

To simplify the relation (2.13), assume that g_{ij} is the identity matrix; later, it will turn out this assumption is not important. To simplify notation in subscripts, let 6 stand for $n-1$ and 7 stand for n, and let a circumflex over repeated indices represent a summation where the index runs over values less than 7. Now solve the relation (2.13):

$$\rho_{i77} = \begin{cases} \frac{1}{2}\rho_{jji} - \rho_{i\widehat{jj}}, & i < 7 \\ \rho_{\widehat{jj}7} - 2\rho_{7\widehat{jj}}, & i = 7. \end{cases}$$

Using these equations, the coefficient of $\omega_{(67)}$ in Θ_i is congruent to

$$\begin{cases} \rho_{i67} - \rho_{i76}, & i < 6 \\ \rho_{766} + \rho_{667} - 2\rho_{\widehat{jj}7}, & i = 6 \\ 2\rho_{\widehat{jj}6} - 2\rho_{666} + \rho_{jj6}, & i = 7 \end{cases}$$

modulo the ρ_{ijk} for $k < 6$. With these extra annihilators, we get

$$s_{n-2} = \binom{n+1}{2} + n + n.$$

From our calculation in the proof of Prop. 2.3, $s_n = 0$ and

$$s_1 + \cdots + s_{n-1} = \operatorname{codim} E = n\binom{n+1}{2} - n + n^2$$

so $s_{n-1} = 2\binom{n+1}{2}$. Now we compute that

$$s_1 + 2s_2 + 3s_3 + \cdots + ns_n = \binom{n+1}{2}^2 - n^2 - \binom{n+1}{2} + n\binom{n+1}{2} - n.$$

Since the space of integral elements of (2.4) satisfying the independence condition form a smooth affine bundle over N with rank precisely this number, by Cartan's Test (III.1.11 in [7]) we conclude that our arbitrary integral n-plane E at a point where $g_{ij} = \delta_{ij}$ is ordinary. However $Gl(n)$ has an obvious action on the fibre of N over U, under which any g_{ij} can be taken to the identity matrix. Since the structure equations (2.5) are $Gl(n)$-invariant, we can conclude that every integral n-plane is ordinary. Thus the system is involutive. In particular, applying the Cartan-Kähler Theorem (III.2.3 in [7]) we conclude

Theorem 2.5 *Through every point in N there exists a smooth analytic integral n-manifold of the system* (2.4) *satisfying the independence condition $\Omega \neq 0$. In particular, every 1-jet of a vector field and 1-jet of a metric satisfying $g^{ij}\Gamma^k_{ij} = 0$ is the 1-jet of an analytic soliton metric defined on a small neighbourhood in U.*

Note that all local solitons are obtained this way, since by Prop. 2.3 they will be analytic.

We can now improve on Prop. 2.3 by noting that involutivity and our calculation above imply that the codimension of the characteristic variety $\Xi^{\mathbb{C}}_E$ is one and the degree is $s_{n-1} = 2\binom{n+1}{2}$ (see Cor. V.3.7 in [7]). Since we proved that $\Xi^{\mathbb{C}}_E$ is contained in the irreducible quadric hypersurface $g^{ij}\xi_i\xi_j = 0$, we deduce that $\Xi^{\mathbb{C}}_E$ must *be* this quadric counted with multiplicity $\binom{n+1}{2}$.

The classical inference that soliton metrics in harmonic coordinates depend on $s_{n-1} = n(n+1)$ arbitrary functions of $n-1$ variables comes from the mechanism in the proof of Cartan-Kähler used to create a sequence of problems to which the Cauchy-Kowalevski theorem applies. After going through the involutivity calculation, one should attempt to find out just what the arbitrary functions are. In fact, the author has calculated that if the vector field X is assumed to be the coordinate vector field $\dfrac{\partial}{\partial x_n}$ then the components g_{ij} of the metric, and their first derivatives in the x_n direction, can be specified along the hyperplane $x_n = 0$ transverse to X; this would account for $n(n+1)$ arbitrary functions along the hyperplane.

Counting identities and the system with fixed metric

When an exterior differential system becomes involutive we know that all its prolongations will also be involutive, and there is a simple way of calculating the Cartan characters of successive prolongations. This in turn enables us to calculate the rank of the bundle of integral n-planes for each prolongation. This gives a count of identities satisfied by solutions, in the following way. In terms of our earlier discussion of involutivity, if we know the rank of the submersion $\mathcal{R}^{(\|+\infty)} \to \mathcal{R}^{(\|)}$, we can calculate the codimension of $\mathcal{R}^{(\|+\infty)}$ in the space of "free" $(k+1)$-jets. We will think of this as the number of identities the $(k+1)$-jets of solutions satisfy. Finally, $\mathcal{R}^{(\|+\infty)}$ is the bundle of integral elements of the contact system on $\mathcal{R}^{(\|)}$, since a possible tangent plane to an integral manifold passing through a point of $\mathcal{R}^{(\|)}$ (i.e. a k-jet of a solution) is a $(k+1)$-jet of a solution.

We are interested in identities that result from imposing the soliton condition in addition to the harmonic coordinate condition, so we will base our comparisons on the latter. The system for a metric g_{ij} with respect to which the coordinates are harmonic lives on $U \times S_+ \times R$ and is generated by the 1-forms

$$\gamma_{ij} = dg_{ij} - \rho_{ijk}\omega^k,$$

where $\rho_{ijk} \equiv g_{ip}d\Gamma^p_{jk} + g_{jp}d\Gamma^p_{ik} \mod \omega^l$ and we have the dependence relation

$$g^{jk}(2\rho_{ijk} - \rho_{jki}) = 0.$$

It is easy to calculate that this system is involutive with characters $s_1 = \ldots = s_{n-1} = \binom{n+1}{2}$, $s_n = \binom{n+1}{2} - n$. These characters will let us count the dimensions of the free k-jets relative to the soliton condition.

In the following table, the entry in the kth row, $k \geq 0$, is the dimension of the bundle of k-jets of solutions over a $(k-1)$-jet solution. To obtain the number of identities at each level we subtract the number in the third column from the sum of the numbers in the first two columns.

	g_{ij} with harmonic coordinates	vector field X	solution to (2.3)	# of identities
0-jets	$\binom{n+1}{2}$	n	$\binom{n+1}{2} + n$	—
1-jets	$n\binom{n+1}{2} - n$	n^2	$n\binom{n+1}{2} - n + n^2$	—
2-jets	$\binom{n+1}{2}^2 - n^2$	$n\binom{n+1}{2}$	$\frac{n(n+1)(n^2+3n-6)}{4}$	$\binom{n+1}{2} + n$
3-jets	$\frac{n^2(n^2-1)(n+4)}{12}$	$n\binom{n+2}{3}$	$\frac{n(n-1)(n^3+7n^2+6n-12)}{12}$	$n\binom{n+1}{2} + n^2 - n$
4-jets	$\frac{n^2(n^2-1)(n+2)(n+5)}{48}$	$n\binom{n+3}{4}$	$\frac{n^2(n-1)(n+7)(n^2+3n-2)}{48}$	$\binom{n+1}{2}^2 + n\binom{n}{2}$

It is easy to see that the 2-jet identities must be

$$X_{ij} + X_{ji} = R_{ij} + \lambda\, g_{ij},$$

which is second-order only in g, together with

$$g^{jk}(X_{ijk} + X_{jik} - X_{jki}) = 0,$$

which is implied by the second Bianchi identity for the covariant derivative of the Ricci tensor.[2] Moreover, involutivity implies that the identities on the 3-jets will be obtained by differentiating the identities on the 2-jets just once — i.e. no further identities will appear by differentiating twice and equating mixed partials.

Suppose, however, that we only want to know what identities there are that are first-order in X, and we do not care how many derivatives of g are involved. Then what we should do is start with a fixed metric g, and find out what integrability conditions arise from (2.2), which is now an overdetermined equation for X.

To this end we will define a differential system on the orthonormal frame bundle $\mathcal{F}$ of g, with canonical forms ω^i and Levi-Civita connection forms ϕ^i_j satisfying the structure equations

$$\begin{aligned} d\omega^i &= -\phi^i_j \wedge \omega^j \\ \phi^j_i &= -\phi^i_j \\ d\phi^i_j &= -\phi^i_k \wedge \phi^k_j + \Phi^i_j \\ \Phi^i_j &= \tfrac{1}{2} R^i_{jkl}\omega^k \wedge \omega^l \end{aligned}$$

On $\mathcal{F} \times \mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}}$ define the forms

$$\eta_i = dX_i - (X_j\phi^j_i + (R_{ij} + \lambda\delta_{ij} + A_{ij})\omega^j), \qquad A_{ji} = -A_{ij}$$

Then $-d\eta_i \equiv X_j\Phi^j_i + (R_{ijk}\omega^k + \pi_{ij}) \wedge w^j \bmod \eta_l$, where

$$\pi_{ij} = dA_{ij} - A_{ik}\phi^k_i - A_{kj}\phi^k_j$$

[2]If we assume that X is a divergence, i.e. $f_{ij} = R_{ij} + \lambda\, g_{ij}$ for some function f, then this identity simplifies to

$$2f_jR^j_i + \nabla_i R = 0,$$

which would determine f up to a constant if the Ricci curvature were non-degenerate.

and $R_{ijk} = \nabla_k R_{ij}$ is the covariant derivative of the Ricci tensor.

We are looking for n-manifolds along which $\Omega \neq 0$ and the 1-forms η_i vanish. The Cartan system for this Pfaffian system is $\{\omega^i, \eta_i, \pi_{ij}\}$. When we try to find integral n-planes we can ignore the 1-forms ϕ^i_j in our coframe which do not appear in this system. (Strictly speaking, the Pfaffian system drops to the quotient $\mathcal{F}/\mathcal{O}(\backslash)$, but we will continue to work with forms on $\mathcal{F}$.) It turns out that there is a unique integral n-plane at each point, given by

$$\pi_{ij}|_E = (R_{kij} - R_{kji} - X_p R^p_{kij})\omega^k.$$

Thus we should add the corresponding 1-forms

$$\theta_{ij} = \pi_{ij} + (X_p R^p_{kij} - R_{kij} + R_{kji})\omega^k$$

to our system. Now the θ_{ij} must be closed modulo the system $\{\eta_i, \theta_{ij}\}$; otherwise, there will be no integral n-planes satisfying $\Omega|_E \neq 0$. In this way, calculating $d\theta_{ij}$ gives us integrability conditions.

The expression of this integrability condition has the following interesting geometric interpretation. Let Θ be the sheaf of germs of Killing fields on a Riemannian manifold, and let D_0 be the first-order linear differential operator taking X to the symmetric bilinear form $X_{ij} + X_{ji}$. In the case where the metric has constant curvature Calabi [8] defined a fine resolution of Θ

$$0 \to \Theta \hookrightarrow \Phi^0 \xrightarrow{D_0} \Phi^1 \xrightarrow{D_1} \Phi^2 \xrightarrow{D_2} \dots$$

where Φ^0 is the sheaf of sections of TM, Φ^1 is the sheaf of symmetric bilinear forms, and Φ^k is the sheaf of sections of the kernel of the map

$$\Lambda^2 T^*M \otimes \Lambda^k T^*M \to T^*M \otimes \Lambda^{k+1} T^*M$$

which anti-symmetrizes on the last $k+1$ factors. (In particular the symmetries of the Riemann curvature tensor imply that it lies in Φ^2.) Gasqui and Goldschmidt have also achieved resolutions of Θ in the case of locally symmetric [14] and conformally flat spaces [15].

For our purposes, define $D_1 : \Phi^1 \to \Phi^2$ as follows: if $s \in \Phi^1$ has components s_{ij}, let

$$D_1(s)_{ijkl} = s_{iljk} - s_{jlik} - s_{ikjl} + s_{jkil} + s_{ip}R^p_{jkl} - s_{jp}R^p_{ikl},$$

where s_{ijkl} are the components of the second covariant derivative of s. (It is clear that $D_1(s)$ is a section of $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$; to see that $D_1(s) \in \Phi^2$, calculate

$$\begin{aligned} D_1(s)_{ijkl}\, \omega^j \wedge \omega^k \wedge \omega^l &= (s_{iljk} - s_{ikjl} - s_{jp}R^p_{ikl})\, \omega^j \wedge \omega^k \wedge \omega^l \\ &= (s_{ijkl} - s_{ijlk} - (s_{jp}R^p_{ikl} + s_{ip}R^p_{jkl}))\, \omega^j \wedge \omega^k \wedge \omega^l = 0 \end{aligned}$$

using the first Bianchi identity, $R^p_{jkl}\omega^j \wedge \omega^k \wedge \omega^l = 0$.) This D_1 coincides with Calabi's operator in the constant curvature case. $D_1 \circ D_0$ is not zero in general, but it is only first-order in X:

$$\begin{aligned} D_1 \circ D_0(X)_{ijkl} = 2(X_p(R^p_{lijk} - R^p_{kijl}) + X_{pk}R^p_{lij} - X_{pl}R^p_{kij}) \\ + (X_{ip} - X_{pi})R^p_{jlk} + (X_{pj} - X_{jp})R^p_{ilk}. \end{aligned}$$

Now, the soliton condition (2.2) can be written as

$$D_0(X) = 2\,\mathrm{Rc} + \lambda\, g$$

and the integrability condition for the exterior differential system with g fixed is

$$D_1(D_0(X)) = D_1(2\,\mathrm{Rc} + \lambda\, g)$$

which is first-order in X but fourth-order in g. The author has calculated that the soliton system for 1-jets of X and 4-jets of g satisfying the above two conditions in involutive. Hence these exhaust the identities implied by the soliton condition that are first-order in X.

Chapter 3

Solitons on Compact Three-Manifolds

The work of Hamilton [19] on the Ricci flow on S^2 might be summarized as

1. prove long-time existence
2. prove convergence to a soliton
3. classify solitons on S^2

The last step is easy in two dimensions since the vector field X must be conformal and a divergence. The only vector field available forces the metric to have an S^1 symmetry, and then it comes down to classifying the solutions to an ODE.

In this chapter we extend the classification of solitons to compact three-manifolds. We show that, in fact, the only solitons are constant curvature metrics.

Solitons and Breathers

In general, a "breather" is a solution to an evolution equation that is periodic over time. For our purposes, it means a solution to the normalized Ricci flow that is periodic, up to diffeomorphism; that is,

$$g_T = \phi^* g_0$$

for some fixed period T and diffeomorphism ϕ. It follows that breathers, like solitons, have uniformly bounded curvature and volume. It will turn out that in dimension three the Ricci flow does not admit any non-trivial breathers either.

Proposition 3.1 *Any soliton or breather on a compact connected manifold is either Einstein with nonpositive Ricci curvature, or has positive scalar curvature.*

The scalar curvature obeys

$$\frac{\partial R}{\partial t} = \Delta R + 2|\,\mathrm{Rt}\,|^2 + \frac{2}{n} R(R - r) \tag{3.3.1}$$

where 'Rt' is the traceless part of the Ricci tensor. Consider a point in space and time where R is at a global minimum. If $R < 0$ there, then $\Delta R \geq 0$ and $\dfrac{\partial R}{\partial t} = 0$ force $\mathrm{Rt} = 0$ and $R = r$ there; hence R is constant over M at this time. By applying the same argument at each point of M at this time, we the metric is Einstein. Otherwise, $R \geq 0$ always.

Assume R has minimum value zero; using (3.1), we know that $\Delta R \leq 0$ at this point. Let Ω be the open set where $\Delta R < 0$ at this time. If Ω is not empty, then R can only achieve its infimum — zero — on the boundary of Ω, and by the Hopf maximum principle the outward normal derivative of R is negative there. This contradicts $dR = 0$ there. Then Ω is empty, and since R achieves its maximum somewhere, R is constant — in fact, identically zero —

by the maximum principle, and the metric is Einstein by the same argument as above.

It follows that solitons and breathers have a positive lower bound for the scalar curvature. Now we can apply the main result of this chapter :

Theorem 3.2 *If g_t obeys the normalized Ricci flow on a compact M^3 and has scalar curvature bounded below by a positive constant R_0 for all $t \geq 0$, then there exists a function $\phi(t) > 0$ such that*

$$\mathrm{Rc}(g_t) \geq -\phi(t)\, g_t$$

and $\lim_{t \to \infty} \phi(t) = 0$.

The proof of this theorem will depend on applying a maximum principle to the curvature tensor. We will actually run the unnormalized flow on the same initial metric, since the corresponding evolution equations for the curvature are simpler and we know how to go back and forth between the two flows. We will show that as the manifold shrinks to a point, the minimum as well as the maximum of the scalar curvature approach infinity. This is important since we will construct a pinching set — that is, a set defined by inequalities on the curvature that are preserved by the flow — that forces the lowest eigenvalue of Rc at a point to go towards zero as $R \to \infty$ at that point.

Theorem 3.3 *If g_t obeys the normalized Ricci flow on compact M^3 and has* $\mathrm{Rc} \geq 0$*, and average scalar curvature r bounded above for all $t \geq 0$, then g_t converges to a metric of constant positive curvature.*

In [18], Hamilton has shown that either the Ricci curvature becomes positive immediately, or M splits locally as a product of a one-dimensional flat factor and a surface with positive curvature, and this splitting is preserved by the flow. In the former case, we know g_t converges to constant positive curvature. To rule out the latter case, consider the evolution equation for r under the normalized flow:

$$\frac{d}{dt}\int R\mu_g = -\int < \frac{\partial g}{\partial t}, \mathrm{Rc} - \frac{1}{2}Rg > \mu_g.$$

(Here we have fixed the volume to be one.) We can calculate the integrand on the right pointwise by diagonalizing the Ricci tensor. We obtain

$$< 2rg/n - 2\,\mathrm{Rc}, \mathrm{Rc} - Rg/2 >= -rR/3$$

and $d/dt\ r = r^2/3$. This would mean r increases without bound, contradicting our assumption.

Corollary 3.4 *There are no three-dimensional solitons or breathers on a compact connected M^3 other than constant curvature metrics.*

The metrics g_t must have r bounded, and by the proposition above, $R > 0$ or else the metric is Einstein, i.e. constant curvature. If $R > 0$ then, by our main result, the lower bound for the lowest eigenvalue of the Ricci tensor goes to zero from below as $t \to \infty$. But since the minimum over M of the lowest eigenvalue is a periodic function of time, it must have been at least zero to begin with. Now Theorem 3.3 completes the proof. In the remainder of this chapter we will prove Theorem 3.2.

Controlling the Scalar Curvature

Suppose s is the time coordinate under the normalized flow, and let g_t on M^n be the unnormalized flow with the same initial data. Then g_s and g_t differ by a change in scale and a reparametrization in time: if $\psi(t) = \mathrm{vol}(g_t)^{-2/n}$ then g_s is given by

$$s = \int_0^t \psi(t)dt$$

$$g_s = \psi(t) g_t.$$

We assume g_s evolves with scalar curvature bounded above $R_0 > 0$. Consider the evolution equation for the volume of g_t:

$$\frac{d}{dt} \log \mathrm{vol}(g_t) = -r(g_t) = -r(g_s)\psi(t) \leq -R_0 \,\mathrm{vol}(g_t)^{-2/n}.$$

Since the integrals of the ordinary differential equation $d/dt\ \log v = -R_0 v^{-2/n}$ go to zero in finite time, we know that $\mathrm{vol}(g_t)$ hits zero at some finite time T. (Note that g_t remains smooth until time T since g_s is smooth.) Since $R(g_t) \geq \psi(t) R_0$, the scalar curvature of g_t goes to infinity everywhere on M at time T.

Some Three-Dimensional Geometry

On a three-dimensional oriented inner product space V, the volume form gives us an isometry between V and $\Lambda^2 V$. Under this isometry, an orthonormal basis (e_1, e_2, e_3) is carried to $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$. Given any quadratic form on $\Lambda^2 V$, we can find an orthonormal basis of V so that the form is diagonalized in the corresponding basis for $\Lambda^2 V$. What this means for Riemannian geometry is that at every point p of a three-dimensional Riemannian manifold (M, g) we can choose an orthonormal basis for $T_p M$ that diagonalizes the curvature tensor; that is,

$$R_{2323} = m_1,\ R_{3131} = m_2,\ R_{1212} = m_3$$

and all other components are zero.

The m_i are the eigenvalues of the curvature operator. It is easy to check that in this frame the Ricci tensor is also diagonalized. We will assume

$$m_1 \le m_2 \le m_3;$$

then $m_1 + m_2 \le m_1 + m_3 \le m_2 + m_3$ are the eigenvalues of the Ricci tensor, and $R = 2(m_1 + m_2 + m_3)$.

The Maximum Principle

Rather than evolving a metric g on the tangent bundle, we will use the "Uhlenbeck trick" (see [18]), that is, evolve a gauge transformation on TM so that g pulls back to be a fixed metric h. If we diagonalize the curvature tensor Rm with respect to g at a point, we also diagonalize the its pullback $\widetilde{\mathrm{Rm}}$ with respect to h. Since the metric on $V = \Lambda^2 T^* M$ is fixed, the following maximum principle applies to $\widetilde{\mathrm{Rm}}$.

Theorem 3.5 (Hamilton [18]) *Let V be a vector bundle on a compact manifold M and h be a fixed metric on V. Suppose g is a metric on M and ∇ a connection on V compatible with h, both possibly varying in time. Let ϕ be a vector field on V tangent to the fibers. Assume X is a closed subset of V, convex in each fibre, invariant under parallel translation at all times, and such that solutions of the ODE $ds/dt = \phi(s)$ for sections of V remain inside X. Then solutions of the heat equation $\partial s/\partial t = \Delta s + \phi(s)$ also remain inside X.*

Fortunately, when we diagonalize the curvature, the right-hand side of the ODE that corresponds to the Uhlenbeck-normalized evolution equation for curvature is also diagonalized. Now we can write down a system of ODE for the eigenvalues of the curvature:

$$\dot{m}_1 = m_1^2 + m_2 m_3$$

$$\dot{m}_2 = m_2^2 + m_1 m_3$$

$$\dot{m}_3 = m_3^2 + m_1 m_2.$$

Since the right-hand side is homogeneous in the m_i, dilation in space and time is a symmetry of this system. This means that the integral curves of this system in $\mathbb{R}^{\nvDash}$ project onto a well-defined set of trajectories on the unit sphere (see Figure 3.1). We will just be interested in the behaviour of the ODE restricted to the wedge defined $m_1 \leq m_2 \leq m_3$ and $m_1 + m_2 + m_3 > 0$. There, the integral curves are attracted either to the line $m_1 = m_2 = m_3$ or to the line $m_1 = m_2 = 0$.

The Pinching Set

Our set X will be given by $m_1 + m_2 \geq -2f(z)$ for a positive function f of $z = m_1 + m_2 + m_3 = R/2$. Since X is defined in terms of eigenvalues relative to h it will be invariant under parallel translation. We will require

1. $f''(z) \leq 0$, in order for X to be convex in each fibre;

2. X to be preserved by the ODE; and

3. $\lim_{z\to\infty} f(z)/z = 0$.

This will then force $(m_1 + m_2)/R$, which is a dilation-invariant quantity, to be at least zero as $R \to \infty$.

To compute condition (2), note that

$$\frac{d}{dt}(m_1 + m_2 + 2f(z)) = (m_1 + m_2)z - m_1m_2 + 2(z^2 - (m_1 + m_2)m_3 - m_1m_2)f'$$

Along the boundary of X given by $m_1+m_2 = -2f(z)$ we need this derivative to be nonnegative. Assume $f' > 0$; then for a fixed z the derivative is minimized when $m_2 = m_1$. Then our condition for f becomes

$$f' \geq \frac{m_1^2 - m_1 z}{z^2 - 2m_1 z + 3m_1^2} = \frac{f(f+z)}{2f^2 + (z+f)^2}$$

By making df/dz larger than necessary we get a simpler ODE for f. Choose

$$f' = \frac{f}{f+z}$$

Solutions of this ODE are invariant under the dilation $f \to \lambda f$, $z \to \lambda z$; and, once f is positive it is an increasing function of z. Thus we can choose f so that $\inf_M m_1 + m_2 \geq -f(\inf_M z)$ for the initial metric g_0, so that the curvature of g_0 lies inside X.

To see that $f'' \leq 0$, let $p = z/f$. Then

$$\frac{dp}{dz} = \frac{1}{f+z} = \frac{p}{p+zp}$$

Since p is monotone increasing, $f' = 1/(p+1)$ is decreasing. Finally, by separation of variables $\log p + p = \log z + C$, so $\lim_{z\to\infty} p = \infty$; thus $f/z \to 0$.

This ends the proof of Theorem 3.2.

Figure 3.1: Trajectories on the unit sphere: the projection is $x = \dfrac{m_1 - m_2}{\sqrt{3}(m_1 + m_2 + m_3)}$, $y = \dfrac{m_1 + m_2}{m_1 + m_2 + m_3} - \dfrac{2}{3}$; the large triangle is Rc ≥ 0, and the small triangle is Rm ≥ 0.

Chapter 4

Deformation Theory of Compact Solitons

In this chapter we investigate the deformation theory of compact Einstein manifolds as compact Ricci solitons. Since Einstein metrics are a kind of trivial compact soliton, the first-order Einstein deformations are a subspace of the first-order soliton deformations. We show that there is a complementary subspace, consisting of genuine soliton deformations, which is isomorphic to a certain eigenspace for the Laplacian. This isomorphism enables us to calculate this subspace for any compact symmetric space, and draw some conclusions about higher-order integrability for these deformations.

Notation and Conventions

M will always be an oriented compact manifold, g a metric on M. On the complex $\mathcal{A}^\bullet(\mathcal{T}^*\mathcal{M})$ of differential forms, the adjoint of d is δ and the Laplace-Beltrami operator is $\Delta = d\delta + \delta d$. On sections of $\otimes^k T^*M$ the *rough Laplacian* is $\nabla^*\nabla$, where $-\nabla^*$ takes a covariant derivative and traces on the last two indices. (In this chapter only we take the "geometer's sign convention", so that the ordinary Laplacian and rough Laplacian have non-negative eigenvalues.) The operator div is the restriction of ∇^* to sections of $S^2(T^*M)$, so it takes a symmetric bilinear form $s_{ij}\omega^i \otimes \omega^j$ to $-g^{jk}s_{ijk}\omega^i$, ω^i being some coframe. Its adjoint, div*, takes $a_i\omega^i$ to $\frac{1}{2}(a_{ij}+a_{ji})\omega^i \otimes \omega^j$, so that $\operatorname{div}^* \alpha = \frac{1}{2}\mathcal{L}_{\alpha^\#}\, g$. For a 1-form α we have the identity $\delta\alpha = -\operatorname{tr}_g \operatorname{div}^* \alpha$ and the Weitzenbock formulas

$$\Delta\alpha - \nabla^*\nabla\alpha = \widetilde{\mathrm{Rc}}(\alpha)$$

$$(\frac{1}{2}\delta d + d\delta)\alpha - \operatorname{div}\operatorname{div}^* \alpha = \widetilde{\mathrm{Rc}}(\alpha)$$

where $\widetilde{\mathrm{Rc}}$ represents the Ricci tensor raised to be an endomorphism of T^*M. (So, for example, $\widetilde{\mathrm{Rc}}$ is a scalar on an Einstein manifold.)

The Ebin Slice Theorem

The set of all metrics on manifold M may be thought of as an open subset in the infinite-dimensional vector space of sections of $S^2T^*(M)$. Of course the diffeomorphisms of M act on this set by pull-back, and we regard as being geometrically distinct only those metrics that do not differ by a diffeomorphism. In order to talk about the metrics near, but distinct from, a given metric g, we need a transversal to the orbits of the diffeomorphism group. This is what the slice theorem gives us, up to the isometries of g.

Let $\mathcal{M}^{\int}$ be the Banach manifold of H^s-metrics on M, i.e. metrics defined almost everywhere on M whose first s derivatives are square-integrable with respect to some fixed norm. If we identify metrics that are a.e. the same, then the Sobolev embedding theorem gives $\mathcal{M}^{\int} \hookrightarrow \mathcal{C}^{\|}(\mathcal{S}^{\in}\mathcal{T}^*)$ for $s \geq k + n/2$. The space $\mathcal{D}^{\int+\infty}$ of H^{s+1}-diffeomorphisms acts on $\mathcal{M}^{\int}$.

Theorem 4.1 (Ebin [13]) *Assume $s > n/2 + 2$. Let g be an H^s-metric on M and I_g the group of H^{s+1}-isometries of g. Then there exists a canonically defined submanifold $\mathfrak{S}^{\mathfrak{s}}$ of $\mathcal{M}^{\int}$ such that*

1. *for any $\phi \in I_g$, $\phi(\mathfrak{S}^{\mathfrak{s}}) \subset \mathfrak{S}^{\mathfrak{s}}$*
2. *if $\gamma \in \mathcal{D}^{\int+\infty}$ is such that $\gamma(\mathfrak{S}^{\mathfrak{s}}) \cap \mathfrak{S}^{\mathfrak{s}}$ is not empty, then $\gamma \in I_g$*
3. *there is a neighbourhood U of the coset I_g in $\mathcal{D}^{\int+\infty}/\mathcal{I}_{\}}$ and a lift of U into $\mathcal{D}^{\int+\infty}$ that makes $U \times \mathfrak{S}^{\mathfrak{s}}$ homeomorphic to a neighbourhood of g in $\mathcal{M}^{\int}$.*

Moreover, the tangent space to $\mathfrak{S}^{\mathfrak{s}}$ at g is the kernel of div_g.

Remark 4.2 *Let $\mathcal{M}^{\int}_{\rfloor} \subset \mathcal{M}^{\int}$ be the set of metrics of a fixed volume c. Then there is also a smooth slice $\mathfrak{S}^{\mathfrak{s}}_{\mathfrak{c}} \subset \mathcal{M}^{\int}$ through g, whose tangent space at g is* $\ker \mathrm{div}_g \bigcap \ker \int \mathrm{tr}$, *where* $\int \mathrm{tr}\, h = \int_M \mathrm{tr}_g\, h\mu_g$.

The slice theorem gives us a reasonable facsimile of the moduli space of metrics near g. Since the slice $\mathfrak{S}$ is the same as the moduli space up to the action of a finite-dimensional Lie group, it is a kind of local pre-moduli space.

The Soliton Pre-Moduli Space

From now on let g be a fixed Einstein metric on M with Einstein constant ε, and let $\mathfrak{S}_{\mathfrak{c}}$ be the slice through g. It has been shown (see [5], p.351) that the set of Einstein metrics in an open neighbourhood of g inside $\mathfrak{S}$ lie in a finite-dimensional smooth submanifold Z_E of $\mathfrak{S}$ and in fact form a *C-analytic* set — that is, a set which is locally defined as the zero locus of a finite number of real-analytic functions. Z_E is constructed in such a way that its tangent space at g is the space of first-order Einstein deformations of g.

To a similar end, on $\mathfrak{S}_\mathfrak{c} \times \mathfrak{H}^{\mathfrak{s}-\mathbf{1}}(\mathfrak{T}^*\mathfrak{M})$, where $H^{s-1}(T^*M)$ is the space of H^{s-1} differential 1-forms, we define

$$\operatorname{Sol}(g,\xi) = \operatorname{Ein}(g) - \operatorname{div}_g^* \xi = \operatorname{Rc}(g) - \frac{r}{n} g - \operatorname{div}_g^* \xi.$$

Then compact solitons near g are exactly the zero locus of Sol.

Theorem 4.3 *There is an open neighbourhood U of $(g,0)$ inside $\mathfrak{S}_\mathfrak{c} \times \mathfrak{H}^{\mathfrak{s}-\mathbf{1}}(\mathfrak{T}^*\mathfrak{M})$ and a smooth finite-dimensional submanifold $Z_S \subset U$ such that the zeros of* Sol *inside U form a C-analytic set that lies in Z_S.*

We will construct Z_S by an implicit function argument. In order to use the Banach space implicit function theorem we will have to find a vector space onto which the differential of Sol at $(g,0)$ surjects. Suppose $h \in T_g\mathfrak{S}_\mathfrak{c}$ and $\eta \in H^{s-1}(T^*M)$; this differential turns out to be

$$\operatorname{Sol}_g^{'}(h,\eta) = F(h) - \operatorname{div}_g^* \eta$$

where, as calculated in [5] and [23], F is an elliptic operator defined by

$$F(h) = \frac{1}{2}\nabla^*\nabla h - \frac{1}{2}\nabla^2 \operatorname{tr} h - \widetilde{\operatorname{Rm}}(h)$$

and

$$\widetilde{\operatorname{Rm}}(h)_{ij} = R_i{}^k{}_j{}^l h_{kl}.$$

Because F is elliptic we have the decomposition $H^{s-2}(S^2T^*) = \operatorname{im} F \oplus \ker F^*$. Since $H^s(S^2T^*) = \ker \operatorname{div} \oplus \operatorname{im} \operatorname{div}^* = T_g\mathfrak{S}_\mathfrak{c} \oplus \mathbb{R}\mathfrak{d} \oplus \operatorname{im} \operatorname{div}^*$ as an orthogonal direct sum — under the L^2 inner product — and $F(\operatorname{div}^* \xi) = \operatorname{div}^* \operatorname{div}(\operatorname{div}^* \xi)$, $F(g) = -\varepsilon g$, then

$$H^{s-2}(S^2T^*) = (F(T_g\mathfrak{S}_\mathfrak{c}) + \operatorname{im}(\operatorname{div}^* \operatorname{div})) \oplus \mathbb{R}\mathfrak{d} \oplus \ker \mathbb{F}^*,$$

where, again, the direct sums are orthogonal. (When $\varepsilon = 0$, the $\mathbb{R}\mathfrak{d}$ term is absorbed into $\ker F^* = (\operatorname{im} F)^\perp$.) Let π_1 be orthogonal projection onto the first direct summand and π_2 be orthogonal projection onto the remainder, which is finite-dimensional. Then $\pi_1 \circ \operatorname{Sol}$ has surjective differential at $(g, 0) \in \mathfrak{S}_\mathfrak{c} \times \mathfrak{H}^{\mathfrak{s}-1}(\mathfrak{T}^*\mathfrak{M})$, and by the implicit function theorem its zero locus Z_S is a smooth submanifold near $(g, 0)$.

The tangent space to Z_S at $(g, 0)$ consists of (h, η) such that

$$\operatorname{div} h = 0, \qquad \int \operatorname{tr} h = 0, \qquad F(h) = \operatorname{div}^* \eta \tag{4.4.1}$$

Define the Bianchi operator[1] $\beta : \Gamma(S^2T^*) \to \Gamma(T^*)$ by

$$\beta(h) = 2 \operatorname{div}(h) + d(\operatorname{tr} h).$$

Then $\beta \circ F = (\Delta - 2\varepsilon) \circ \operatorname{div}$, so the above equations imply $\beta(\operatorname{div}^* \eta) = 0$. By a Weitzenbock formula, $\beta \circ \operatorname{div}^* = \Delta - 2\varepsilon$, so η is a 2ε-eigenvector for the Laplacian. Since for any given η the solution space for $F(h) = \operatorname{div}^* \eta$ is finite-dimensional by ellipticity, it follows that the tangent space to Z_S at $(g, 0)$ is finite-dimensional.

Finally, $\pi_2 \circ Sol|_{Z_S}$ is clearly analytic in g and ξ, and its image lies in a finite-dimensional vector space, so its zero locus inside Z_S is a C-analytic set.

When $F(h) = 0$, h is just a first-order Einstein deformation of g, and when $\operatorname{div}^* \eta = 0$, η is dual to a Killing field of g. Hence we will call such solutions of

[1] We give this operator this name since the second Bianchi identity for the Ricci tensor can be expressed as $\beta(\operatorname{Rc}(g)) = 0$

the first-order soliton deformation equations (4.1) *E-K deformations.* If these are the only first-order soliton deformations of g, we say g is *relatively infinitesimally non-deformable* (R∞ND). If there are actually no soliton metrics near g that are not Einstein, we say g is *relatively rigid.*

It may happen that g is R∞ND but not relatively rigid, i.e. there may be a curve through g in the space of solitons but is tangent to the space of Einstein deformations at g. However, there are special cases where an inference can be made:

Proposition 4.4 *If an Einstein metric g admits no Einstein deformations and is R∞ND, then it is relatively rigid. In particular, S^n is relatively rigid.*

By assumption the only solutions to (4.1) are $h = 0$ and η being dual to a Killing field of g. In particular Z_S has the same dimension as the space of Killing fields, and thus Z_S is filled out by Killing fields.

For S^n sitting in $n+1$-dimensional Euclidean space, the Einstein constant is $\varepsilon = n-1$ and the eigenvalues of the Laplacian on functions are 0, n, $2(n+1)$, etc. (see, for example, II.4 in [9]). Then by Prop. 4.6 below, the only soliton deformations of S^n are E-K deformations. By Theorem 12.30 in [5], these must have $\mathrm{tr}_g\, h = 0$. Since S^n has constant curvature, $F(h) = \frac{1}{2}\nabla^*\nabla h$; so $F(h) = 0$ implies h is parallel. Since the holonomy of S^n acts irreducibly on the tangent space of any point, and h is traceless, h must be zero. We should point out that rigidity of S^n as a soliton also follows from the pinching results obtained for the Ricci flow in [21] and [24].

The First-Order Deformation Space

Theorem 4.5 *Suppose (M, g) is a compact Einstein manifold with Einstein constant ε. When $\varepsilon \leq 0$ the only first-order soliton deformations are E-K deformations. When $\varepsilon > 0$ and $n > 2$ the space of solutions (h, η) of the first-order soliton deformation equations* (4.1) *at* $(g, 0)$ *splits as an orthogonal direct sum of the E-K deformations and deformations of the form*

$$h = \nabla^2 f + \varepsilon f g, \qquad \eta = -\frac{\varepsilon(n-2)}{2} df$$

where f is a 2ε-eigenfunction on (M, g).

As seen in the proof of 4.3, the deformation equations imply $(\Delta - 2\varepsilon)\eta = 0$; since δ commutes with the Laplace-Beltrami operator, $\delta\eta$ is a 2ε-eigenfunction. Now let $<, >$ indicate the L^2 inner product for tensors. Then by a Weitzenbock formula,

$$< \operatorname{div}^* \eta, \operatorname{div}^* \eta >=< \operatorname{div} \operatorname{div}^* \eta, \eta >= \frac{1}{2} < (\Delta - 2\varepsilon)\eta + d\delta\eta, \eta >= \frac{1}{2} < \delta\eta, \delta\eta > .$$

Since $\delta\eta$ integrates to zero, $\varepsilon \leq 0$ implies $\delta\eta = 0$, which implies $\operatorname{div}^* \eta = F(h) = 0$, and (h, η) is an E-K deformation.

Now assume $\varepsilon > 0$, and let $\mathcal{A}^{\checkmark}_{\lambda}$ be the λ-eigenspace for the Laplacian on p-forms. Suppose $\theta \in \mathcal{A}^{\infty}_{\in\varepsilon}$ is orthogonal to $d(\mathcal{A}'_{\in\varepsilon})$; then $0 =< \theta, d\delta\theta >=< \delta\theta, \delta\theta >$ and the previous calculation show that $\operatorname{div}^* \theta = 0$. Together with $\beta \circ \operatorname{div}^* = \Delta - 2\varepsilon$, this shows that $\mathcal{A}^{\infty}_{\in\varepsilon} = \mathcal{A}'_{\in\varepsilon} \oplus \ker \operatorname{div}^*$ as an orthogonal direct sum. Thus we can always arrange that $\eta = df$ by subtracting out the duals of Killing fields.

Once $\eta = df$, tracing $F(h) = \operatorname{div}^* \eta$ gives $(\Delta - \varepsilon) \operatorname{tr} h = -\delta\eta$. This implies $\delta\eta + \varepsilon \operatorname{tr} h \in \mathcal{A}'_{\varepsilon}$. But by a theorem of Lichnerowicz (see 12.30 in [5]) the lowest positive eigenvalue of Δ on functions is at least $n\varepsilon/(n-1)$, so $-\varepsilon \operatorname{tr} h = \delta\eta = 2\varepsilon f$. Thus we are led to try a solution of the form

$$h = A\nabla^2 f + Bfg, \qquad \eta = df$$

for some constants A and B.

Next we calculate $\nabla^*\nabla(\nabla^2 f)^2$:

$$\begin{aligned} g^{kl} f_{ijkl} &= g^{kl}(f_{klij} + f_{k[il]j} + f_{ki[jl]} + f_{[ik]jl} + f_{i[jk]l}) \\ &= g^{kl}(f_{klij} + (f_p R^p_{kil})_j + f_{pi} R^p_{kjl} + f_{kp} R^p_{ijl} + (f_p R^p_{ijk})_l) \\ &= -\nabla^2 \Delta f + 2\varepsilon \nabla^2 f - 2\widetilde{\mathrm{Rm}}(\nabla^2 f) + f_p g^{kl} R^p_{ijkl} \end{aligned}$$

Since on an Einstein manifold, $g^{jl} R_{ijkl} = \varepsilon g_{ik}$ is parallel, then $0 = g^{jm}(R_{ijkml} + R_{ijmlk} + R_{ijlkm})$ shows $g^{jm} R_{ijklm} = 0$. Then $0 = g^{jm}(R_{ijklm} + R_{ikljm} + R_{iljkm})$ shows $g^{jm} R_{ikljm}$ is symmetric in k and l, which implies $g^{jm} R_{ikljm} = 0$.

Thus $\nabla^*\nabla(\nabla^2 f) = 2\widetilde{\mathrm{Rm}}(\nabla^2 f)$. We also calculate $\nabla^*\nabla(fg) = 2\varepsilon fg = 2\widetilde{\mathrm{Rm}}(fg)$. If we plug the above guess into the deformation equations (4.1), we get

$$\operatorname{div} h = A(\Delta - \varepsilon) df - B df = 0 \quad \Longrightarrow B = \varepsilon A$$

$$\operatorname{tr} h = (-2\varepsilon A + nB) f \quad \Longrightarrow \int \operatorname{tr} h = 0$$

$$F(h) - \operatorname{div}^* \eta = -\frac{1}{2}(n-2)\varepsilon A \nabla^2 f - \nabla^2 f$$

This determines A and B, and we obtain the form in the statement of the theorem by rescaling f. The direct sum decomposition now follows from the linearity of (4.1).

[2] Here, square brackets around indices indicates an alternating sum of the corresponding tensor components

By Richard Hamilton's work [19], we know the only soliton on S^2 is the constant curvature metric. Nevertheless for completeness we include the following

Proposition 4.6 *On S^n, $n \geq 2$, with the standard metric g of constant curvature +1, the only first-order soliton deformations are E-K deformations.*

By the proof of 4.5 we can assume $\eta = df$ and $\operatorname{tr}_g h = -2f$. So write $h = k - \frac{2}{n} fg$, where k is traceless. Then $F(h) - \operatorname{div}^* \eta = \frac{1}{2}\nabla^*\nabla k - \widetilde{\mathrm{Rm}}(k)$. Since we have constant curvature, this is $\frac{1}{2}\nabla^*\nabla k + k$. Since the rough Laplacian $\nabla^*\nabla$ can only have non-negative eigenvalues, $k = 0$. Then $0 = \operatorname{div} h = -\frac{2}{n} df$ implies $f = 0$ as well.

Next we will consider some Einstein manifolds where the first-order deformation spaces can be realized in a concrete fashion.

First-Order Deformations of Symmetric Spaces

Suppose $M = G/K$ is a compact Einstein symmetric space with $\varepsilon > 0$. Then it follows that G is a compact semisimple Lie group and a bi-invariant metric on G such that the projection $\pi : G \to M$ is a Riemannian submersion must be a negative multiple of the Killing form. (In fact, if we use the metric which is -1 times the Killing form B, the Einstein constant ε on G (and M) is exactly 1/2.)

Under a bi-invariant metric the left cosets gK in G are totally geodesic, so it follows that the Laplacian on functions commutes with pullback via π (see [29]). The group G acts on $C^\infty(M, \mathbb{C})$ and $C^\infty(G, \mathbb{C})$ by pullback: $g \cdot f = (L_{g^{-1}})^* f$. If we use the $-B$ metric on G, the Laplacian exactly coincides with the Casimir operator for $C^\infty(G, \mathbb{C})$ as a $\mathfrak{g}$-module.

For compact symmetric spaces, there is a one-to-one correspondence between (G-irreducible components of) eigenspaces of the Laplacian on $M = G/K$ and irreducible G-modules that are "of class 1" with respect to K — that is, modules which admit a line fixed by K. Indeed, let V be the module and $v \in V$ a vector fixed by K, and let $<,>$ be a G-invariant inner product on V. Then to a vector $w \in V$ we associate the function

$$\varphi_w(gK) =< w, g \cdot v >$$

which is well-defined on M. The map $\varphi : V \to C^\infty(M, \mathbb{C})$ is equivariant and nonzero — since $\varphi_v(K) =< v, v >$ — so is injective. The fact that the G-modules V fill out all of $C^\infty(M, \mathbb{C})$ follows from Theorem 8.1 in [29].

So, in order to enumerate symmetric spaces admitting non-trivial soliton deformations, we should figure out what compact semisimple Lie groups have irreducible representations with Casimir operator +1.

Lemma 4.7 (Koiso [23], 5.1, 5.2) *If $\mathfrak{g}$ is a simple complex Lie algebra of compact type then the Casimir operator for any $\mathfrak{g}$-module V is greater than 1/3. Moreover, the Casimir operator is +1 if and only if $V = \mathfrak{g}$, the adjoint representation.*

Suppose G/K decomposes as a product of irreducible symmetric spaces G_i/K_i. (We can pass to a finite cover, if necessary, to do this, and not lose any eigenfunctions.) It follows from the first assertion of the lemma that a G-module with Casimir +1 comes from at most two irreducible factors, and, if two, neither can be of type II. If $V = V_1 \otimes V_2$, where V_i is an irreducible G_i-module, then one module, say V_1, has Casimir operator at most $\varepsilon = 1/2$. By the theorem of Lichnerowicz, the Einstein manifold G_1/K_1 cannot have eigenvalue ε, so V_1 cannot be class 1 with respect to K_1. When one considers how V splits as a sum of irreducible K-modules, it is clear that V cannot have a fixed line. Thus we only need to consider irreducible symmetric spaces; all other symmetric space deformations with break up as sums of deformations on the irreducible factors.

When (M, g) is a compact irreducible symmetric space of type I, $\mathcal{A}'_{\in\varepsilon}$ is nonzero iff M is hermitian and the corresponding representation is the adjoint representation. (This follows from the second assertion of the lemma: if $\mathfrak{g}$ has a line fixed by K, then that line must lie in the Lie algebra of K; so K is not semisimple, which is equivalent, by Cor. 8.7.10 in [30], to G/K being hermitian.) When (M, g) is of type II, $\mathcal{A}'_{\in\varepsilon}$ is nonzero only when M is G_2, G is $G_2 \times G_2$, and V is tensor product of two copies of the seven-dimensional G_2-module (see p. 655 in [23]).

In [23], Koiso used these facts to calculate the space of Einstein deformations for symmetric spaces of compact type. In the table that follows, we show how our results compare. For example, we obtain deformations for $\mathbb{CP}^{\ltimes}$, $n \geq 2$, while Koiso shows that $\mathbb{CP}^{\ltimes}$ is rigid as an Einstein manifold. This is due to the fact that for h to be an infinitesimal Einstein deformation, the trace of h must be identically zero. However, in the case of products like $\mathbb{CP}^{\ltimes} \times \mathbb{CP}^{\Bbbk}$, Koiso overcomes this restriction by letting $h = \nabla^2 f_1 + \varepsilon f_2 g_2$, where the f_i are 2ε-eigenfunctions from $\mathbb{CP}^{\ltimes}$ and g_2 is the metric on $\mathbb{CP}^{\Bbbk}$.

space	Einstein-deformable	soliton-deformable	dim M	dim $\mathcal{A}'_{\in\varepsilon}$
$\frac{SU(n)}{SO(n)}$, $n > 2$	yes	no		
$\frac{SU(2n)}{Sp(n)}$, $n > 2$	yes	no		
$\frac{SU(p+q)}{S(U(p)\times U(q))}$	if $p, q \geq 2$	if $p+q \geq 3$	$2pq$	$(p+q)^2 - 1$
$\frac{SO(p+2)}{SO(p)\times SO(2)}$	no	yes	$2p$	$\binom{p+2}{2}$
$\frac{SO(p+q)}{SO(p)\times SO(q)}$	no	no		
$SO(2n)/U(n)$	no	yes	$n(n-1)$	$\binom{2n}{2}$
$Sp(n)/U(n)$	no	yes	$n(n+1)$	$2n^2 + n$
$\frac{E_6}{SO(10)U(1)}$	no	yes	32	78
E_6/F_4	yes	no		
$\frac{E_7}{E_6U(1)}$	no	yes	54	135
other type I	no	no		
$SU(n)$, $n > 2$	yes	no		
G_2	no	yes	14	49
other type II	no	no		

Higher-Order Integrability

We will begin to consider the higher-order integrability by reviewing some standard material on integrability and obstructions to integrability (see also [5], p.348f).

Suppose we have a smooth curve $g(t)$ through $g = g(0)$ in a Banach space and E is a smooth map to another Banach space. Let

$$h_k = (\frac{d}{dt})^k g(t)|_{t=0}$$

$$E_g^k(h_1, \dots h_k) = (\frac{d}{dt})^k E(g)|_{t=0}$$

(The second definition makes sense since the right-hand side only depends on the first k derivatives of $g(t)$.) Note that E_g^k is polynomial in the h_i and linear in h_k. A formal power series $g + \sum \frac{t^k}{k!} h_k$ at g is said to be a formal solution to $E(g(t)) = 0$ if $E(g) = 0$ and $E_g^k(h_1, \dots h_k) = 0$ for all k. A first-order deformation h_1 is formally integrable if it can be extended to a formal solution.

If the differential $E_g' = E_g^1$ is surjective then formal integrability is automatic for any h_1. This is because we can calculate that $E_g^k(h_1, \dots h_k) = E_g^1(h_k) + P_g^k$, where P_g^k depends only on $h_1, \dots, h_{k-1}$. (In particular, P_g^2 is a quadratic in h_1.)

If E_g^1 does not surject, then suppose some linear operator B_g, depending smoothly on g, kills the image of E_g^1. If we have solved $E_g^j(h_1, \dots, h_j) = 0$ for $j < k$, then $B_g(P_g^k(h_1, \dots, h_{k-1})) = 0$. Thus the quotient $\ker B_g / \operatorname{im} E_g^1$ is the obstruction space for extending our power series solution.

- In the case of deformations of complex structures, E_J^1 is $\bar{\partial} : \Lambda^{0,1}(T^{1,0}M) \to \Lambda^{0,2}(T^{1,0}M)$, B_J is $\bar{\partial}$, and the obstruction space is isomorphic to $H^2(M, \Theta)$, where Θ is the sheaf of germs of holomorphic vector fields.
- In the case of deformations of Einstein metrics, the obstruction space is isomorphic to $\ker E_g^1$, the space of first-order Einstein deformations.

Proposition 4.8 *In the case of soliton deformations of Einstein metrics, the complement of the image of* $T_g\mathcal{M}^\int \times \mathcal{H}^{\int-\infty}(\mathcal{T}^*\mathcal{M})$ *under the differential of* Sol *is*

$$(\operatorname{im} \mathrm{Sol}_g')^\perp = \mathrm{EID} \oplus \beta^*(d(\mathcal{A}'_{\in\varepsilon})) \oplus \mathbb{R}\eth,$$

where EID is the space of Einstein deformations of g, β^* is the adjoint of β, and $\mathbb{R}\eth$ are the multiples of g.

In [23], Koiso shows that $\operatorname{im}(\operatorname{Ein}_g'|_{\ker\int\operatorname{tr}}) \oplus \mathrm{EID} = \ker\beta \bigcap \ker\int\operatorname{tr}$. Since $\int\operatorname{tr}\circ\operatorname{Sol} = 0$ then $\int\operatorname{tr}\circ\operatorname{Sol}_g' = 0$, so we can't have multiples of g in the image. Since $\operatorname{Ein}_g'(g)$ is at most a multiple of g, we can use Koiso's result to see that

$$(\operatorname{im}\operatorname{Sol}')^\perp = \left(\mathrm{EID} \oplus \left(\ker\beta \bigcap \ker\int\operatorname{tr}\right)^\perp\right) \bigcap \ker\operatorname{div} + \mathbb{R}\eth.$$

Now, $(\ker\beta \bigcap \ker\int\operatorname{tr})^\perp = \operatorname{im}\beta^* + \mathbb{R}\eth$ since β is an underdetermined elliptic operator. Since $\beta = 2\operatorname{div} + d\circ\operatorname{tr}$, then $\beta^* = 2\operatorname{div}^* + g\delta$. It is easy to see that $\operatorname{im}\beta^*$ is disjoint from the multiples of g: if $\beta^*(\xi) = cg$, then tracing gives $(n-2)\delta\xi = cn$, which implies $d\delta\xi = 0 \implies \delta\xi = 0 \implies c = 0$ if $n \neq 2$, or $c = 0$ immediately if $n = 2$. Furthermore, $\operatorname{div}\circ\beta^*(\xi) = (\Delta - 2\varepsilon)\xi$. Since on an Einstein manifold we have $\mathcal{A}^\infty_{\in\varepsilon} = \ker\operatorname{div}^* \oplus \lceil(\mathcal{A}'_{\in\varepsilon})$ and $\beta^*(\ker\operatorname{div}^*) = 0$, the above decomposition follows.

As was just observed, we can at least take $B_g = \int\operatorname{tr}_g$. If it happens that EID is zero then $d(\mathcal{A}'_{\in\varepsilon})$ is the obstruction space; for example, second-order integrability depends on whether or not $P_g^2(h_1)$ has any component in $d(\mathcal{A}'_{\in\varepsilon})$ — which we cannot hit.

In the case of symmetric spaces, the above table gives many examples where the space EID is zero. Moreover, the quadratic P_g^2, restrictied to first order deformations h_1 coming from $V = \mathcal{A}'_{\in\varepsilon}$, will be a G-module map from S^2V to sections of S^2T^*M. If S^2V has no component isomorphic to V, then $P_g^2(h_1)$ must be orthogonal to $d(\mathcal{A}'_{\in\varepsilon})$. By checking how S^2V decomposes as a direct sum of irreducible G-modules, we obtain

Proposition 4.9 *Any first-order soliton deformation is integrable to second order in the case of* $SO(p+2)/SO(p) \times SO(2)$*,* $SO(2n)/U(n)$*,* $Sp(n)/U(n)$*,* $E_6/SO(10)U(1)$*,* $E_7/E_6U(1)$*, and* G_2*.*

This argument does not apply to the complex Grassmannians because, for $G = SU(n)$, $n \geq 3$, the module $S^2\mathfrak{g}$ always has an irreducible component isomorphic to $\mathfrak{g}$. This does not necessarily imply that the deformations are not integrable to second order. That has to be checked. The case of $\mathbb{CP}^{k}$ is the most tractible, since the gradient of a 2ε-eigenfunction is a holomorphic vector field, and the holomorphic sectional curvature is constant. However, even this calculation might prove to be rather long — cf. [23], p. 650-652, 663-667.

On the other hand, suppose M is one of the symmetric spaces to which the above proposition applies, and let V be the G-module giving rise to the 2ε-eigenfunctions. Since we have to solve the equation $-E^1_g(h_2) = P^1_g(h_1)$, and E^1_g is linear, we can assume that h_2 is drawn from submodules of the space of sections of S^2T^*M isomorphic to the irreducible components of S^2V. Now, $P^2_g(h_1, h_2)$ is a sum of a cubic term in h_1 and a bilinear term in h_1 and h_2. If $V \otimes S^2V$ has no components isomorphic to V, then $P^2_g(h_1, h_2)$ will be orthogonal to the obstruction space and we can integrate to third order. In fact, we can assume that h_3 is drawn from $V \otimes S^2V$. The induction step can be stated as

Proposition 4.10 *Let* M *and* V *be as above. If* $E^k_g(h_1, \dots, h_k) = 0$ *and* h_1 *is drawn from* V*,* h_2 *from* S^2V*,* h_3 *from* $V \otimes S^2V$*, and so forth, and* $(\otimes^{k-1}V) \otimes S^2V$ *has no components isomorphic to* V*, then there is an* h_{k+1} *drawn from* $(\otimes^{k-1}V) \otimes S^2V$ *such that* $E^{k+1}_g(h_1, \dots, h_{k+1}) = 0$*.*

In view of the rapid growth of $(\otimes^{k-1}V)\otimes S^2V$, it seems unlikely that this calculation would show integrability to any high order. If it turned out to do so, it would be a remarkable fact in itself and would provide us with some evidence for formal integrability of the soliton deformations for that space.

Chapter 5

Constructions of Complete Solitons via Warped Products

Let $(M^n, d\sigma^2)$ be a compact Einstein manifold with Einstein constant $\varepsilon > 0$. Let $d\theta^2$ denote the standard metric of constant curvature $+1$ on S^k, $k \geq 1$. On $\mathbb{R}^{k+1} \times \mathbb{M}$, with radial coordinate t, consider the doubly-warped product metric

$$ds^2 = dt^2 + f(t)^2 d\theta^2 + g(t)^2 d\sigma^2, \tag{5.5.1}$$

where f and g are some functions of the radial coordinate. We wish to construct Ricci soliton metrics with this special form. Since M could be a symmetric space, and S^k has a lot of symmetry, it makes sense that the Ricci flow should only move in the radial direction. Matters being so, let us assume that the vector field X points in the radial direction, and is in fact the gradient of a function $h(t)$. Then it only remains to solve the equation $\nabla^2 h = \mathrm{Ric}(ds^2)$.

As we will see below, this condition leads to a system of second-order ODE for f, g, and h. We will show that there exists a 1-parameter family of solutions which give rise to complete Ricci metrics on $\mathbb{R}^{k+1} \times \mathbb{M}$. In trying to find the right solutions it will be helpful to "mod out" by the symmetries of the ODE;

our problem will essentially be reduced to examining the trajectories of a vector field in a phase space of dimension four (dimension three if $k = 1$). We will eliminate a large number of solutions using the following criteria for the metric (5.1) to be smooth near the copy of M lying over the origin in $\mathbb{R}^{n+k}$:

Proposition 5.1 ([4], §8.9) *Assume f, g are smooth positive functions defined on an interval $(0, r)$. Then ds^2 extends to give a smooth positive definite metric on a neighbourhood of $t = 0$ if and only if*

1. *$f(t)$ extends smoothly to an odd function of t with $f'(0) = 1$*
2. *$g(t)$ extends smoothly to a strictly positive, even function of t.*

The ODE and its first integral

First we will derive the ODE from the structure equations for the metric (5.1). Let ω^i be a local orthonormal coframe on M and let η^α be a local orthonormal coframe on S^k. Then dt, $f\eta^\alpha$, $g\omega^i$ is an orthonormal coframe for (5.1). If the matrix-valued Levi-Civita connection forms for ω^i and η^α are W, N respectively, satisfying $d\omega = -W \wedge w$ and $d\eta = -N \wedge \eta$, then the structure equations for the new metric are

$$d \begin{pmatrix} dt \\ f\eta \\ g\omega \end{pmatrix} = - \begin{pmatrix} 0 & -f'\,{}^t\eta & -g'\,{}^t\omega \\ f'\eta & N & 0 \\ g'\omega & 0 & W \end{pmatrix} \wedge \begin{pmatrix} dt \\ f\eta \\ g\omega \end{pmatrix}.$$

(In this chapter, let a prime denote differentiation by t.) These structure equations give the Levi-Civita connection forms; after calculating the curvature forms one can obtain the components of the Ricci tensor with respect to the (new) coframe:

$$R_{11} = -n\frac{g''}{g} - k\frac{f''}{f}$$

$$R_{\alpha\beta} = \left(-n\frac{f'g'}{fg} - \frac{f''}{f} + (k-1)\left(\frac{1-(f')^2}{f^2}\right)\right)\delta_{\alpha\beta}$$

$$R_{ij} = \left(\frac{\epsilon}{g^2} - (n-1)\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - k\frac{f'g'}{fg}\right)\delta_{ij}.$$

Using the fact that h is a function of t alone, we compute

$$\nabla^2 h = h''dt^2 + ff'h'd\theta^2 + gg'h'd\sigma^2.$$

Hence $\nabla^2 h = \mathrm{Ric}(ds^2)$ amounts to

$$\begin{cases} h'' = -n\dfrac{g''}{g} - k\dfrac{f''}{f} \\ f'' = -n\dfrac{f'g'}{g} - h'f' + (k-1)\left(\dfrac{1-(f')^2}{f}\right) \\ \dfrac{g''}{g} = \dfrac{\varepsilon}{g^2} - (n-1)\left(\dfrac{g'}{g}\right)^2 - k\dfrac{f'g'}{fg} - h'\dfrac{g'}{g}. \end{cases} \tag{5.5.2}$$

In order to use Prop. 5.1 we have to know when solutions extend to smooth odd and even functions of t. In this matter we have

Proposition 5.2 *Suppose f, g, h are smooth solutions of* (5.2) *for $t > 0$, and*

$$\lim_{t\to 0^+} f = 0, \qquad \lim_{t\to 0^+} g > 0, \qquad \lim_{t\to 0^+} f' = 1,$$

and g'/f and $(k-1)\left(\frac{1-(f')^2}{f^2}\right) - \frac{h'f'}{f}$ are bounded for t near zero. Then f, g, h extend to smooth odd, even and even functions, respectively, satisfying (5.2).

By hypothesis we can extend f, g, h to odd, even and even functions which are C^1 at $t = 0$; then it is easy to verify that they are solutions for $t < 0$ as well. If (5.2) is re-written as a first-order system, as $t \to 0$ the right-hand side approaches a finite, nonzero limit. (For example, the t-derivative of f approaches one.) If we think of our solution as a trajectory in a six-dimensional space (with coordinates f, g, h, f', g', h'), the trajectory approaches a point where the vector field is nonzero. Thus our solution curve must coincide with the smooth integral curve through this point guaranteed by standard ODE theory.

The system (5.2) has a first integral which follows from the Bianchi identity. Namely, if we substitute $\nabla^2 h = \mathrm{Ric}$ into the formula $g^{ik}(h_{ijk} - h_{ikj}) = h_p R^p_j$, and use the second Bianchi identity $2g^{ij}R_{ikj} = \nabla_k R$ for the Ricci tensor, we get $d(\Delta h) = d((h')^2)$, where the Laplacian has non-negative spectrum. If we eliminate the second derivatives using (5.2), this first integral implies

$$2kn\frac{f'g'}{fg}+n(n-1)\left(\frac{g'}{g}\right)^2-\frac{n\varepsilon}{g^2}+2nh'\frac{g'}{g}+2kh'\frac{f'}{f}+(h')^2-k(k-1)\left(\frac{1-(f')^2}{f^2}\right)=C$$

for some constant C.

From now on we will assume that $\varepsilon = 1$, since we can still scale the Einstein metric on M as much as we like using the function g.

Modding out by symmetries

In the ODE (5.2), t and h do not not appear. Thus the symmetries include translation in t and h, but we can also scale f, g and t simultaneously by the same positive factor. The following quantities are invariant under these symmetries:

$$x = g', \qquad z = g\frac{f'}{f}, \qquad y = gh' + nx + kz, \qquad w = \frac{g}{f}.$$

(When $k = 1$, we can also scale in f independently of the other coordinates on the jet space, so only x, y, and z are invariant in this case.) The ODEs implied by (5.2) for the new coordinates are

$$\begin{cases} g\dfrac{dx}{dt} = 1 - xy + x^2 \\ g\dfrac{dy}{dt} = x(y - nx) - kz^2 \\ g\dfrac{dz}{dt} = z(x - y) + (k-1)w^2 \\ g\dfrac{dw}{dt} = w(x - z) \end{cases} \tag{5.5.3}$$

The quantities x, y, z, w were chosen to simplify the first integral: it becomes

$$y^2 - nx^2 - kz^2 - k(k-1)w^2 = Cg^2 + n$$

In particular, the two-sheeted hyperboloid $y^2 - nx^2 - kz^2 - k(k-1)w^2 = n$ is a stable set for (5.3): integral curves that start inside the hyperboloid stay in the hyperboloid.

Because of the factor g, (5.3) does not give a well-defined dynamical system in x, y, z, w-space; rather, we have a Pfaffian system

$$\frac{dx}{1 - xy + x^2} = \frac{dy}{x(y - nx) - kz^2} = \frac{dz}{z(x - y) + (k-1)w^2} = \frac{dw}{w(x - z)}.$$

Given an integral curve of this differential system, we can obtain a solution to the original ODE (5.2) by the quadratures

$$\frac{dg}{g} = \frac{xdx}{P(x,y)},$$

$$dt = \frac{gdx}{P(x,y)},$$

$$dh = \frac{(y - nx - kz)dx}{P(x,y)}$$

where $P(x,y) = 1 - xy + x^2$, and we obtain $f(t)$ by $f = g/w$. (When $k = 1$ and w is not available, we have to integrate $df/f = z\,dx/P(x,y)$.)

In order for the corresponding metric to extend smoothly across the origin in $\mathbb{R}^{⌉+⊮}$, by 5.2 we need $w \to \infty$, $z \to \infty$, $x \to 0$, and $y \to \infty$ as $t \to 0$ on the curve. In fact, we can compute that

$$\frac{y}{w} = \frac{gh' + ng' + kgf'/f}{g/f} \longrightarrow k.$$

In order to examine what happens to integral curves at "infinity", we will change coordinates to

$$Y = \frac{\sqrt{n}}{y} \qquad X = \sqrt{n}\frac{x}{y}$$

$$Z = \sqrt{k}\frac{z}{y} \qquad W = \sqrt{k(k-1)}\,\frac{w}{y}$$

In these coordinates we compute

$$\begin{cases} \dfrac{g}{y}\dfrac{dX}{dt} = X(X^2 + Z^2 - 1) + \alpha Y^2 \\[2ex] \dfrac{g}{y}\dfrac{dY}{dt} = Y(X^2 + Z^2 - \alpha X) \\[2ex] \dfrac{g}{y}\dfrac{dZ}{dt} = Z(X^2 + Z^2 - 1) + \beta W^2 \\[2ex] \dfrac{g}{y}\dfrac{dW}{dt} = W(X^2 + Z^2 - \beta Z), \end{cases} \tag{5.5.4}$$

where $\alpha = 1/\sqrt{n}$ and $\beta = 1/\sqrt{k}$. We could again regard this as a Pfaffian system in the variables X, Y, Z, W, but instead we will choose a new time coordinate, and replace $\dfrac{g}{y}\dfrac{dX}{dt}$ by $\dfrac{dX}{ds}$, etc. In these coordinates the stable hyperboloid becomes the sphere $X^2 + Y^2 + Z^2 + W^2 = 1$, and in fact we have Liapunov function:

$$\frac{d}{ds}(X^2 + Y^2 + Z^2 + W^2 - 1) = 2(X^2 + Y^2 + Z^2 + W^2 - 1)(X^2 + Z^2).$$

Ricci-flat metrics and radially symmetric solitons

We will motivate our main calculation by looking at integral curves of (5.3) and (5.4) lying in stable sets of low dimension.

Setting $h' = 0$ in $\nabla^2 h = \mathrm{Ric}$ gives a Ricci-flat metric, and this corresponds to $y = nx + kz$ in our first set of invariant coordinates. It is easy to verify that the intersection of the stable hyperboloid with the hyperplane $y = nx + kz$ is also stable. (To get a compact picture, we will usually think instead in terms of the stable unit sphere in the X, Y, Z, W coordinates, and the hyperplane $X/\alpha + Z/\beta = 1$.) Now, when $k = 1$ we do not have the invariant coordinate w; but the system we get in the invariant coordinates x, y, z is the same as the restriction of (5.3) to the stable hyperplane $w = 0$. Thus when $k = 1$ we can intersect these three hypersurfaces to get a single integral curve that gives rise to a Ricci-flat metric on an open subset of $\mathbb{R}^{\mathbb{k}} \times \mathbb{M}$. (In fact, it is not difficult to verify that on the two-sphere given by $W = 0$ and $X^2 + Y^2 + Z^2 = 1$, every integral curve is given by the intersection of the sphere with a 2-plane through the critical points $X = \alpha$, $Y = \pm\sqrt{1 - \alpha^2}$, $Z = 0$, $W = 0$.) If we take the portion of the integral curve with $Y > 0$ and $Z > 0$, and integrate the above quadratures, it turns out that $t \to 0$ near the "zenith" point $X = 0$, $Y = 0$, $Z = 1$, while $t \to \infty$ near the equator. In fact this gives the complete Ricci-flat metric discovered by Bérard-Bergery ([4], §§9.5, 9.6) that is known as the Riemannian Schwarzschild metric (cf. 9.118 in [5]).

Another stable set is the plane obtained by setting Z and W to zero. Integral curves in this plane give product metrics on $S^1 \times M_1$, where M_1 is a warped product of M over an interval. Since f is constant these metrics cannot close up in accordance with Prop. 5.2, but we can look at the case when $M = S^n$ and g goes to zero at one end of the interval. The condition that the soliton metric on M_1 can be smoothly extended to $\mathbb{R}^{\ltimes+\nVdash}$ has been studied (for $n = 2$) by Bryant [6]. The phase portrait for the system 5.4 restricted to the X, Y-plane looks like Figure 5.1. Bryant showed that the separatrix that runs from the

Figure 5.1:

critical point with $X = \alpha$ and $Y = \sqrt{1 - \alpha^2}$ to the center of the disc gives rise to a complete radially symmetric soliton on $\mathbb{R}^{\nVdash}$. The dimension n appears only as a parameter in the system, and in fact the same separatrix gives a complete soliton on $\mathbb{R}^{n+1}$ for all $n \geq 2$.

Now consider how the integral curves for the Schwarzschild metric and the radially symmetric soliton look in the same picture (Figure 5.2) in X, Y, Z-space. Since the soliton curve has $\int dt$ unbounded as it approaches the centre

Figure 5.2:

of the sphere, while the Schwarzschild metric closes up nicely as the curve approaches the zenith of the sphere, it makes sense to look for a curve that interpolates between these two desirable boundary conditions. At least in the case $k = 1$, this was the motivation for what follows.

An interpolating solution

For simplicity we will deal with the case $k > 1$ first. By Prop. 5.2 we want $Y \to 0$, $X \to 0$, $f' = \sqrt{k-1}\frac{Z}{R} \to 1$ and $R \to \sqrt{1-\beta^2}$. These values for X, Y, Z, W give a critical point of (5.4); if we linearize about this point, we find the Jacobian of (5.4) has two positive and two negative eigenvalues. Thus in the unstable manifold of this critical point there is a one-parameter family of curves that should give rise to metrics that close up nicely near the origin in $\mathbb{R}^{\neg+\nVdash}$.

Lemma 5.3 *Assume $k > 1$. Let $\mathfrak{S}$ denote the one-parameter family of integral curves of* (5.4) *that lie in the unstable manifold of the point $X = 0$, $Y = 0$, $Z = \beta$, $R = \sqrt{1-\beta^2}$, and lie strictly in the interior of the sphere $X^2 + Y^2 + Z^2 + W^2 = 1$, and in the half-space $Y > 0$. Then curves in $\mathfrak{S}$ give rise to solutions of* (5.2) *that fulfil the conditions of Prop. 5.2.*

We will begin by establishing the limiting value of X/Y^2 as we approach the critical point along a curve in $\mathfrak{S}$. From the equation

$$\frac{dX}{ds} = X(X^2 + Z^2 - 1) + \alpha Y^2$$

we see that $X \geq 0$ along this curve, since the last term is positive; similarly,

$$\frac{d}{ds}((1+\beta^2)X - \alpha Y^2) = (X^2 + Z^2 - 1)((1+\beta^2)X - \alpha Y^2) + \alpha Y(Y(1+\beta^2) + \alpha X - 1)$$

shows that $\dfrac{X}{Y^2} \leq \dfrac{\alpha}{1+\beta^2}$ for X, Y near zero, since the last term becomes negative. Now that we know X/Y^2 is bounded in absolute value,

$$2\frac{d/ds\ X}{d/ds\ Y^2} = \frac{X}{Y^2}\left(\frac{X^2 + Z^2 - 1}{X^2 + Z^2 - \alpha X}\right) + \frac{\alpha}{X^2 + Z^2 - \alpha X} \tag{5.5.5}$$

shows that X/Y^2 has the limiting value $\alpha/(1+\beta^2)$.[1]

Since $X \sim \alpha/(1+\beta^2)\ Y^2$,

$$\frac{dg}{g} = \frac{X(Y dX - X dY)}{Y(X^2 - X/\alpha + Y^2)} \sim \frac{\alpha^2}{\beta^2(1+\beta^2)} Y dY. \tag{5.5.6}$$

[1]This follows from a recursive form of l'Hôpital's Rule: if $f'(x)/g'(x) = q(x)\, f(x)/g(x) + p(x)$, and f/g is bounded as $x \to a$, and $\lim q(x)$ is finite and different from 1, then $\lim f/g = \lim p/(1-q)$.

So by integration we can obtain $g \to c > 0$ as $Y \to 0$ along the curve. Next,

$$\frac{dt}{g} = \frac{YdX - XdY}{X^2 - X/\alpha + Y^2} \sim \frac{\alpha}{c\beta^2} dY, \tag{5.5.7}$$

so we can arrange that $t \to 0^+$. Since $f = \dfrac{g}{w} = \dfrac{\alpha\sqrt{1-\beta^2}gY}{\beta^2 W}$, $f' = \dfrac{z}{r} = \dfrac{\sqrt{1-\beta^2}Z}{\beta W}$, and $\dfrac{gg'}{f} = xw = \dfrac{\beta^2 XW}{\alpha\sqrt{1-\beta^2}Y^2}$, all the requirements of Prop. 5.2 follow immediately except for the last one. We calculate that

$$(k-1)g^2\left(\frac{1-(f')^2}{f^2}\right) - g^2\frac{h'f'}{f} = (k-1)(w^2 - z^2) - w(y - nx - kz) = nxw$$

$$+\left(\frac{\beta^3 W}{\sqrt{1-\beta^2}} - (1-\beta^2)Z\right)\left(\frac{Z-\beta}{\alpha^2 Y^2}\right) + \beta(\beta W + \sqrt{1-\beta^2}Z)\left(\frac{W - \sqrt{1-\beta^2}}{\alpha^2 Y^2}\right).$$

To see that $(Z-\beta)/Y^2$ and $(W - \sqrt{1-\beta^2})/Y^2$ are bounded as $s \to -\infty$ on the curves in question, we will have to work harder. First of all, for any $\lambda > 0$,

$$\begin{aligned} \frac{d}{ds}(Z - \beta - \lambda Y^2) &= (X^2 + Y^2 - 1)(Z - \beta - \lambda Y^2) \\ &\quad + \lambda Y^2(2\alpha X - X^2 - Z^2 - 1) + \beta(X^2 + Z^2 + W^2 - 1), \end{aligned}$$

and this shows that $Z - \beta - \lambda Y^2 \leq 0$ on these curves. Then $Z - \beta \leq 0$ implies $W - \sqrt{1-\beta^2} \leq 0$, by examining the tangent plane to the unstable manifold at the critical point. Then, when $(Z-\beta)/Y^2$ and $(W - \sqrt{1-\beta^2})/Y^2$ are non-positive, there exist positive constants k_1, k_2, k_3 such that for $s < s_0$,

$$\begin{aligned} \left(-\frac{d}{ds}\right)\left(\frac{Z-\beta}{\alpha^2 Y^2}\right) &= (1 + X^2 + Z^2 - 2\alpha X - \beta(Z+\beta))\left(\frac{Z-\beta}{\alpha^2 Y^2}\right) \\ &\quad - \beta(W + \sqrt{1-\beta^2})\left(\frac{W - \sqrt{1-\beta^2}}{\alpha^2 Y^2}\right) - \beta\frac{X^2}{Y^2} \end{aligned}$$

$$\geq (1-\beta^2+\beta e^{k_1 s})\left(\frac{Z-\beta}{\alpha^2 Y^2}\right)-\beta(2\sqrt{1-\beta^2}-e^{k_2 s})\left(\frac{W-\sqrt{1-\beta^2}}{\alpha^2 Y^2}\right)-\beta e^{k_3 s}$$

$$\left(-\frac{d}{ds}\right)\left(\frac{W-\sqrt{1-\beta^2}}{\alpha^2 Y^2}\right)=(\beta Z+Z^2+X^2-2\alpha X)\left(\frac{W-\sqrt{1-\beta^2}}{\alpha^2 Y^2}\right)$$
$$-\sqrt{1-\beta^2}\left(Z\left(\frac{Z-\beta}{\alpha^2 Y^2}\right)+\frac{X^2}{Y^2}\right)$$

$$\geq 2\beta^2\left(\frac{W-\sqrt{1-\beta^2}}{\alpha^2 Y^2}\right)-\sqrt{1-\beta^2}\left((\beta-e^{k_1 s})\left(\frac{Z-\beta}{\alpha^2 Y^2}\right)+e^{k_3 s}\right)$$

Since the exponential terms die out as $s \to -\infty$, and the solutions of the linear ODE for two variables (obtained by ignoring the exponentials) have $(Z-\beta)/Y^2$ and $(W-\sqrt{1-\beta^2})/Y^2$ bounded for as long as they are both negative, we are done.

Lemma 5.4 *Let $\mathfrak{S}$ be as in Lemma 5.3. Then as $s \to \infty$ curves in the set $\mathfrak{S}$ tend to the center of the sphere $X^2+Y^2+Z^2+W^2=1$ exponentially, and give rise to metrics with $\int dt$ unbounded as $s \to \infty$.*

Let $L = X^2+Y^2+Z^2+W^2$; then $d/ds\, L = 2(X^2+Z^2)(L-1)$, so L is strictly decreasing for curves inside the sphere. But since $L<1$, $d/ds\, L \geq 2L(L-1)$, so $L \to 0$ exponentially. Hence, curves approach the center of the sphere exponentially.

First we will show that X/Y^2 is bounded on these curves, and has limiting value α. Since

$$\frac{d}{ds}\frac{X}{Y^2}=\alpha-\frac{X}{Y^2}(X^2+Z^2-2\alpha X+1)$$

we see that once X/Y^2 is non-negative it stays non-negative; combined with the limiting value obtained in Lemma 5.3, this means $X/Y^2 \geq 0$. Since the curves approach the origin only exponentially, there exists a $k > 0$ such that

$$\frac{d}{ds}\left(\frac{X}{Y^2} - \alpha\right) \leq \left(\frac{X}{Y^2} - \alpha\right)(e^{-ks} - 1) + \alpha e^{-ks}.$$

By solving the corresponding one-variable ODE, we see that $X/Y^2 - \alpha$ stays bounded. We can apply the mean value theorem[2] to (5.5) to get

$$\frac{\alpha}{2(X^2 + Z^2 - \alpha X)} = \delta\left\{\frac{X}{Y^2}\right\} + \left(1 - \frac{X^2 + Z^2 - 1}{2(X^2 + Z^2 - \alpha X)}\right)\frac{X}{Y^2}.$$

Multiplying through by $X^2 + Z^2 - \alpha X$ and taking the limit shows $\lim_{s\to\infty} X/Y^2 = \alpha$.

Next we need to know that Z/Y^2 is bounded: to see this, note that

$$\frac{d}{ds}\log\left(\frac{\beta W}{\alpha Y}\right) = \alpha X - \beta Z \leq e^{-ks}$$

for some k. It follows that W/Y is bounded. Then since

$$\frac{d}{ds}\left(\frac{Z}{W^2}\right) = \beta + \frac{Z}{W^2}(2\beta Z - X^2 - Z^2 - 1)$$

we can show Z/W^2 is bounded by the same argument just used for X/Y^2.

This will enable us to find the limiting value of $(X - \alpha Y^2)/Y^4$ — i.e. the next term in the power series of X. First we calculate that

$$\frac{d}{ds}\left(\frac{X - \alpha Y^2}{Y^4}\right) = (-1-3(X^2+Z^2)+4\alpha X)\left(\frac{X - \alpha Y^2}{Y^4}\right) - \frac{\alpha}{Y^2}(X^2+Z^2-2\alpha X).$$

[2] In the proof of the recursive l'Hôpital's rule, one applies the mean value theorem to get $f'(c)(g(x) - g(a)) = g'(c)(f(x) - f(a))$ for some c between x and the limiting value a. Using the special form assumed for f'/g', one gets

$$p(c) = \delta\left\{\frac{f}{g}\right\} + (1 - q(c))\frac{f(c)}{g(c)},$$

where $\delta\{F\} = F(x) - F(c)$. The rule then follows by letting $x \to a$.

When $(X - \alpha Y^2)/Y^4 > 0$ there are constants C, k_1, k_2 such that

$$\frac{d}{ds}\left(\frac{X - \alpha Y^2}{Y^4}\right) \leq (-1 + e^{-k_1 s})\left(\frac{X - \alpha Y^2}{Y^4}\right) + 2\alpha^2 + Ce^{-k_2 s}$$

and when $(X - \alpha Y^2)/Y^4 < 0$ we similarly have

$$\frac{d}{ds}\left(\frac{X - \alpha Y^2}{Y^4}\right) \geq (-1 + e^{-k_1 s})\left(\frac{X - \alpha Y^2}{Y^4}\right) + 2\alpha^2 - Ce^{-k_2 s}.$$

Solving the corresponding one-variable ODEs shows that $(X - \alpha Y^2)/Y^4$ is bounded. Finally,

$$\frac{d/ds\ (X - \alpha Y^2)}{d/ds\ Y^4} = \left(\frac{X - \alpha Y^2}{Y^4}\right)\left(\frac{-1}{4(X^2 + Z^2 - \alpha X)}\right) - \frac{\alpha}{2Y^2} + \frac{X(X^2 + Z^2)}{4Y^4(X^2 + Z^2 - \alpha X)}.$$

Applying the mean value theorem argument again, one obtains $\lim_{s \to \infty}(X - \alpha Y^2)/Y^4 = 2\alpha^3$.

Now we will check the quadratures (5.6) and (5.7). Using the last result, we find that

$$\frac{dg}{g} \sim -\frac{dY}{Y}$$

and so there is a constant $c > 0$ such that $gY \to c$. Then

$$\frac{dt}{g} \sim -\frac{dY}{\alpha Y^2}$$

shows that $t \to \infty$ as $Y \to 0$.

We will now turn to the special case $k = 1$; hence, we will restrict our attention to the restriction of (5.4) to the hyperplane $W = 0$.

If we linearize (5.4) about the zenith of the sphere, the Jacobian has two positive eigenvalues and one zero eigenvalue. The first two eigenvectors span the YZ plane, while the zero eigenvalue corresponds to moving in the direction parallel to the X-axis; in fact this is the tangent direction to circle of critical points of (5.4), given by the intersection of the sphere with the $Y = 0$ plane. On the other hand, the Center Manifold Theorem (see [3] §4.2) asserts that there is a smooth two-dimensional unstable manifold S swept out by integral curves $X(s), Y(s), Z(s)$ that exponentially approach the zenith as $s \to -\infty$.

Lemma 5.5 *Assume $k = 1$. Let $\mathfrak{S}$ denote the one-parameter family of integral curves of* (5.4) *that lie in the unstable manifold of the point $X = 0$, $Y = 0$, $Z = 1$, and lie strictly in the interior of the sphere $X^2 + Y^2 + Z^2 = 1$, and in the half-space $Y > 0$. Then curves in $\mathfrak{S}$ give rise to solutions of* (5.2) *that fulfil the conditions of Prop. 5.2.*

As in 5.3 we obtain $X/Y^2 \to \alpha/2$, $g \to c > 0$, and $t \to 0^+$ as $s \to -\infty$. In order to compute df/f, we will need to know that $(Z-1)/Y^2$ is bounded on the curves we are interested in. Since the curves approach the zenith exponentially, and using the fact that $Z \leq 1$, there exist k_1, k_2 such that

$$\begin{aligned}\left(-\frac{d}{ds}\right)\left(\frac{Z-1}{Y^2}\right) &= (X^2 + Z^2 - Z + 2\alpha X)\left(\frac{Z-1}{Y^2}\right) - \frac{X^2}{Y^2} \\ &\geq e^{k_1 s}\left(\frac{Z-1}{Y^2}\right) - e^{k_2 s}\end{aligned}$$

for s tending to $-\infty$. By comparing with a one-variable ODE, we conclude that $(Z-1)/Y^2$ is bounded below.

Now we have

$$\frac{df}{f} = \frac{Z(YdX - XdY)}{\alpha Y(X^2 - X/al + Y^2)} \sim \frac{dY}{Y}$$

so that $f \sim c_1 Y$ for some $c_1 > 0$. Since $f' = \dfrac{fZ}{\alpha g Y}$, we can choose c_1/c to arrange that $f' \to 1$. Then since $\dfrac{gg'}{f} = \dfrac{g}{f}\left(\dfrac{X}{Y}\right)$ and $\dfrac{gh'}{f} = \dfrac{1-Z}{\alpha f Y} - \dfrac{X}{\alpha^2 f Y}$, the rest of the requirements of Prop. 5.2 can be verified immediately.

Lemma 5.6 *Let $\mathfrak{S}$ be as in Lemma 5.5. Then as $s \to \infty$ curves in the set $\mathfrak{S}$ tend to the center of the sphere $X^2+Y^2+Z^2 = 1$ exponentially, and give rise to metrics with $\int dt$ unbounded as $s \to \infty$.*

The proof of Lemma 5.4 works here as well, except for getting Z/Y^2 to be bounded. For this we just use

$$\frac{d}{ds}\frac{Z}{Y^2} = \frac{Z}{Y^2}(2\alpha X - X^2 - Z^2 - 1) \leq -\frac{Z}{Y^2}(1 - e^{-ks})$$

for some k and s large, and compare with the solution of the one-variable ODE.

Theorem 5.7 *There exists a one-parameter family of smooth complete Ricci soliton metrics of the form (5.1) on $M \times \mathbb{R}^{2+k}$, where M is a compact Einstein manifold of positive scalar curvature and $k \geq 1$. These metrics have positive Ricci curvature away from $t = 0$. Metrics in a given family are not equivalent under diffeomorphisms that preserve the submersion onto $\mathbb{R}^{2+k}$.*

Existence and completeness follows from the previous four lemmas. Recall that

$$\mathrm{Ric} = \nabla^2 h = h'' dt^2 + ff'h' d\theta^2 + gg'h' d\sigma^2.$$

We calculate that $gh'/y = 1 - X/\alpha - Z/\beta$ and

$$\frac{d}{ds}(1 - X/\alpha - Z/\beta) = (X^2 + Z^2 - 1)(1 - X/\alpha - Z/\beta) + 1 - X^2 - Y^2 - Z^2 - W^2.$$

This shows that $h' > 0$ for $t > 0$, and from our calculations above we know $g' = x > 0$. To see that $h'' > 0$, we calculate that

$$\frac{g^2 h''}{y^2} = \frac{X}{\alpha} + \frac{Z}{\beta} + W^2 - X^2 - Y^2 - (\frac{2}{\beta^2} - 1)Z^2.$$

Let Q stand for the quantity on the right; then

$$\frac{d}{ds}Q = (X^2 + Z^2 - 1)Q + (X^2 + (\frac{2}{\beta^2} - 1)Z^2)(1 - X^2 - Y^2 - Z^2 - W^2)$$
$$+ 2(\frac{1}{\beta^2} - 1)Y^2 Z^2 + 2R^2((1 - Z/\beta)^2 + X^2)$$

shows that $Q > 0$ for $t > 0$.

Given one of these metrics, the origin in the base $\mathbb{R}^{k+1}$ is the point over which the fibres have least volume. The coordinate t measures distance in $\mathbb{R}^{k+1}$ from the origin. The Einstein constant for the fibres is $1/g(t)$, and the k-dimensional volume of the spheres in $\mathbb{R}^{k+1}$ at distance t from the origin is $f(t)$ up to a universal constant depending on k. When $\mathrm{Ric} > 0$, the value of $h'(t)$ is also recoverable from the curvature. Thus the geometric data uniquely determine the integral curve of (5.4), and the last assertion of the theorem is proved.

Remark 5.8 *Since M can be an arbitrary Einstein manifold with $\varepsilon > 0$, we do not have much control over the sectional curvature of the metrics constructed. In particular, these solitons do not necessarily fall into the domain of Hamilton's Harnack inequality [20], which assumes non-negative curvature operator.*

Remark 5.9 *One might generalize our construction (at least in the $k = 1$ case) in the following way: rather than using $M \times \mathbb{R}^2$, let the underlying manifold be a non-trivial complex line bundle over M, where M is now assumed* *to have an almost complex structure for which the Einstein metric is hermitian and the Kähler form is closed. We assume the unit circle bundle has a Yang-Mills connection, and warp this over an interval to get the metric on the line bundle. (This is inspired by the construction of Page-like complete Einstein metrics — see [5], §9.K.) The corresponding ODE for solitons lives naturally in a five-dimensional phase space. Metrics that close up smoothly near the origin of the fibre and are complete correspond to integral curves lying in the intersection of the unstable manifold of one point in phase space and the stable manifold of another point. If these manifolds intersect transversely, their intersection is a single curve, giving the Ricci-flat metric obtained by Bérard-Bergery [4]. So, it seems unlikely that new soliton metrics arise in this way.*

Remark 5.10 *For the radially symmetric solitons on* $\mathbb{R}^{\ltimes}$*,* $n \geq 3$*, obtained by Bryant, the largest eigenvalue of the curvature falls off like the reciprocal of the distance from the origin and the volume of balls grows only like the radius to the exponent* $(n+1)/2$*. So these metrics do not satisfy the hypotheses of Shi's main theorem [27]. (They had better not, since Shi proves convergence of the Ricci flow to a flat metric.) In fact, we conjecture that these solitons are "attractors", in the sense that if an initial metric on* $\mathbb{R}^{\ltimes}$ *has similar curvature and volume growth, perhaps the Ricci flow converges to a symmetric soliton. A first step would be to study the Ricci flow for radially symmetric metrics, which comes down to a non-linear one-dimensional heat equation.*

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Biography

The author was born in 1963 in Port Dover, Ontario, Canada. As a child, he wanted to become an architect or a musician; his interest in mathematics came later on. He received his Bachelor of Mathematics, with joint honours in Applied Math and Computer Science, from the University of Waterloo in 1987 and his Master of Arts degree from Duke University in 1989. He has been supported by a 1967 Graduate Fellowship from the Natural Sciences and Engineering Research Council of Canada and by a James B. Duke Fellowship from Duke University.