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Spectral stability analysis for periodic traveling wave solutions of NLS and CGL perturbations

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Abstract

We consider cnoidal traveling wave solutions to the focusing nonlinear Schrödinger equation (NLS) that have been shown to persist when the NLS is perturbed to the complex Ginzburg–Landau equation (CGL). We show that while these periodic traveling waves are spectrally stable solutions of NLS with respect to periodic perturbations, they are unstable with respect to bounded perturbations. Furthermore, we use an argument based on the Fredholm alternative to find an instability criterion for the persisting solutions to CGL.

Keywords: Evans function; Spectral stability; Nonlinear Schrödinger equation; Complex Ginzburg–Landau equation; Lax pair; Baker eigenfunctions

1. Introduction

The complex Ginzburg–Landau equation (CGL) is used in several contexts such as chemistry, fluid dynamics, and optics [3,26,27]. Furthermore, CGL was shown to be the generic equation modeling the slowly varying amplitudes of post-critical Rayleigh–Bénard convection [31]. CGL is not in general an integrable equation. However, a successful strategy when it comes to the study of existence and stability of solutions is to consider CGL as a perturbation of the nonlinear Shrödinger equation (NLS). One can then take advantage of the fact that NLS is completely integrable. The drawback of that method is that the coefficients of the dissipative terms in CGL must remain small.

In the case of localized traveling wave solutions of CGL such as pulses and holes, this strategy has been implemented in several instances (see for example [21,22,28]). In the case of periodic traveling waves, it has recently been shown [10–12] that perturbations of classes of NLS solutions persist as CGL solutions. The main purpose of this paper is to study the stability of one class of such solutions, namely the cnoidal waves defined in (3). We will restrict our attention to spectral stability with respect to perturbations that have the same period T as the waves, which amounts to restricting the spatial domain of the solution to an interval of length T.

Our results are two-fold. In the first part of the paper, we give a complete description of the set of zeros of the Evans function for the linearization of NLS about these cnoidal waves. To define this function, we insert an ansatz with exponential dependence on time into linearized NLS, obtaining a linear ordinary differential operator $\mathcal{L}_0$. The zeros of the Evans function are the *point spectrum* of $\mathcal{L}_0$, i.e., the set of $\ell \in \mathbb{C}$ for which $\mathcal{L}_0\boldsymbol{\omega} = \ell\boldsymbol{\omega}$ has a nontrivial T-periodic solution. In particular, this point spectrum does not have eigenvalues with positive real part (in fact, it consists of a countably infinite number of points on the imaginary axis) and that implies that cnoidal solutions are linearly stable with respect to perturbations of period T. We also obtain the continuous spectrum of $\mathcal{L}_0$ (i.e., the ℓ-values for which there is a nontrivial $\boldsymbol{\omega}$ bounded on the real line) and show that it intersects the right side of the complex plane, implying that cnoidal wave solutions (3) are unstable when considered on the whole line. (This is consistent with

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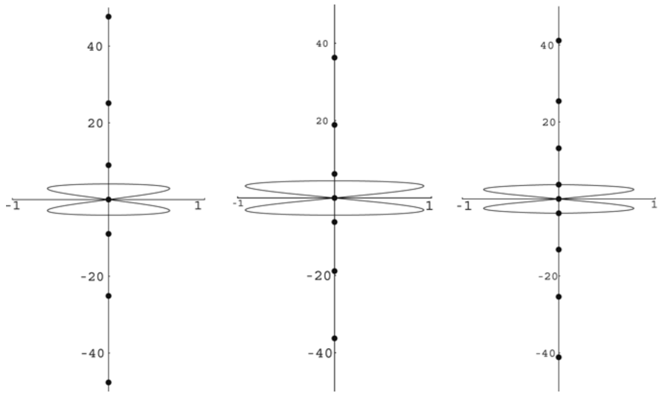

Fig. 1. The dots represent the zeros of the Evans function $M(0, \ell)$ in the complex ℓ-plane, for the values of the modulus $k = .6$, $k = .8$ and $k = .92$, respectively. Note that all zeros are of multiplicity two, except for the origin, which is of multiplicity four. The continuous spectrum consists of the figure eight curve and part of the imaginary axis.

the fact that complex double points in the Floquet spectrum of the Lax pair usually give rise to instabilities [13,14,29], and it is also consistent with recent work [9] in which (3) is numerically shown to be unstable.) These results are summarized in Theorems 1 and 2. Fig. 1 shows the location of the point spectrum (corresponding to the zeros of the Evans function) and the continuous spectrum in the complex ℓ-plane for three values of the modulus of the cnoidal wave solutions of NLS.

The main tool we use is the fact that solutions to linearized NLS can be constructed using squared AKNS eigenfunctions [14, 30], for which explicit formulas are available [5,7]. The main difficulty lies in the fact that one has to show that all the solutions of the eigenvalue problem are obtained that way; in fact, this is not true for the zero eigenvalue. By using an alternate route to construct solutions, and employing the technique of the Evans function (as defined in [17]), we show that the zero eigenvalue has algebraic multiplicity four.

Of course, integrability properties have been used before for determining the spectra of linear operators arising when considering soliton solutions [23,25]. The novelty of our work lies in the fact that we do this for periodic solutions. Two recent works by Gallay and Hărăgus are closely related to the results obtained in this paper concerning the NLS. In [15], families of complex periodic wave solutions are proven to be orbitally stable with respect to complex-valued perturbations with modulus having the same period as the modulus of the periodic wave. However, in the particular case of cnoidal waves, which is the case studied here, the condition on the modulus of the perturbations implies that the perturbations themselves have to be anti-periodic on half a period (see [15], Remark 2 of the Introduction). We do not impose such a restriction on the perturbations. Furthermore, while the main tools used in [15] are based on the work of Grillakis, Shatah, and Strauss [18,19] concerning Hamiltonian formalism, the derivations of our results are mainly based on the fact that NLS has a Lax pair; also, while the results of [15] on the orbital stability implies that the point spectrum is restricted to the real axis, our Theorem 1 describes more precisely the location of the point spectrum. In the second work [16], the continuous spectrum is obtained for small periodic wave solutions of NLS, including the cnoidal case. We do not make the assumption that the waves are small.

In the second part of the paper, we use the knowledge of the point spectrum of $\mathcal{L}_0$ to carry out a perturbative study of the corresponding spectrum for linearized CGL. More precisely, we linearize about the persisting periodic solution of CGL, and we use the Evans function technique and the Fredholm alternative to identify a criterium for linear instability. Under these conditions, eigenvalues of the linearized CGL bifurcate from the origin to the right side of the complex plane as NLS is perturbed to CGL. These results are summarized in Theorem 4 and illustrated in Figs. 2 through 5.

We now briefly describe how this paper is arranged. In Section 2, we define the concept of Evans function for the linearization of NLS about a cnoidal traveling wave. We then determine the spectrum of $\mathcal{L}_0$, stating Theorems 1 and 2, and proving Theorem 3. In Sections 3.1 and 3.2, we describe the persisting solution of CGL and the corresponding linearization. In Section 3.3, we use the concept of Evans function and the Fredholm alternative to study eigenvalues emerging from the origin when NLS is perturbed. The proofs of Theorems 1 and 2 are given in Appendix A, and the analyticity of dependence of the eigenvalues on the perturbation parameter is established in Appendix B.

2. Evans function for an NLS periodic traveling wave

2.1. Linearized NLS and operator $\mathcal{L}_0$

The nonlinear Schrödinger system (see, e.g., [5], Eq. (4.1.1)) is

$$\begin{aligned} &\mathrm{i}q_{1t} + q_{1xx} + 2q_1^2 q_2 = 0 \\ &-\mathrm{i}q_{2t} + q_{2xx} + 2q_1 q_2^2 = 0. \end{aligned} \tag{1}$$

If (q_1, q_2) is a solution of (1) that satisfies the *reality condition* $q_2 = \overline{q_1}$, then q_1 satisfies the standard focusing NLS equation. (Similarly, if $q_2 = -\overline{q_1}$ then q_1 satisfies defocusing NLS.)

We make a change of variables $q_1 = \mathrm{e}^{-\mathrm{i}\alpha t} u_1$, $q_2 = \mathrm{e}^{\mathrm{i}\alpha t} u_2$, for a real constant α, and obtain the following system:

$$\begin{aligned} &\mathrm{i} u_{1t} + \alpha u_1 + u_{1xx} + 2u_1^2 u_2 = 0 \\ &-\mathrm{i} u_{2t} + \alpha u_2 + u_{2xx} + 2u_1 u_2^2 = 0. \end{aligned} \tag{2}$$

This system has several T-periodic traveling wave solutions of the form $u_1 = \overline{u_2} = U_0(x - ct)$ where U_0 is expressible in terms of elliptic functions. However, these only persist as T-periodic solutions of the complex Ginzburg–Landau (CGL) equation when $c = 0$ [10–12]. We limit our attention here to the real-valued solution of (2) given by

$$u_1(x, t) = u_2(x, t) = U_0(x), \qquad U_0(x) = \delta k \operatorname{cn}(\delta x; k), \tag{3}$$

with $\alpha = \delta^2(1 - 2k^2)$ where k is the elliptic modulus, $0 < k < 1$. The period of this solution is $T = 4K/\delta$, where $K = K(k)$ is the complete elliptic integral of the first kind.

We then linearize the system (2) about the solution (3). Substituting $u_1 = U_0 + v_1$, $u_2 = U_0 + v_2$ in (2) and discarding higher-order terms in the v's, we get

$$\begin{aligned} &\mathrm{i} v_{1t} + \alpha v_1 + v_{1xx} + 4U_0^2 v_1 + 2U_0^2 v_2 = 0 \\ &-\mathrm{i} v_{2t} + \alpha v_2 + v_{2xx} + 4U_0^2 v_2 + 2U_0^2 v_1 = 0. \end{aligned} \tag{4}$$

To understand the stability of the stationary solution (3) of (2), we examine solutions of the linearization that have exponential dependence on time. Thus, we substitute into (4) the ansatz

$$v_1 = \mathrm{e}^{\ell t} w_1(x), \qquad v_2 = \mathrm{e}^{\ell t} w_2(x), \tag{5}$$

where w_1 and w_2 are assumed to be independent of time t, obtaining the following ODE system for w_1, w_2

$$\begin{aligned} &\mathrm{i}\ell w_1 + \alpha w_1 + w_{1xx} + 4U_0^2 w_1 + 2U_0^2 w_2 = 0, \\ &-\mathrm{i}\ell w_2 + \alpha w_2 + w_{2xx} + 4U_0^2 w_2 + 2U_0^2 w_1 = 0, \end{aligned}$$

which can be written as the eigenvalue problem

$$\mathcal{L}_0 \boldsymbol{\omega} = \ell \boldsymbol{\omega}, \qquad \boldsymbol{\omega} = (w_1, w_2)^{\mathrm{T}}, \tag{6}$$

where $\mathcal{L}_0$ is a linear operator given by

$$\mathcal{L}_0 = J\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \begin{pmatrix} 4U_0^2 + \alpha & 2U_0^2 \\ 2U_0^2 & 4U_0^2 + \alpha \end{pmatrix}\right], \qquad J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{7}$$

Let $L^2(\mathbb{R})$ be the space of square integrable functions. The point spectrum of $\mathcal{L}_0$ over the space $\left(L^2(\mathbb{R})\right)^2$ is empty (i.e., (6) has no nontrivial solutions whose components are in $(L^2(\mathbb{R}))^2$) and the spectrum is purely continuous, consisting of those values of ℓ for which (6) has a nontrivial bounded solution [17]. In the rest of the article, we refer to this set as the *continuous spectrum of* $\mathcal{L}_0$. Over the space $(L_2(T))^2$ of two-dimensional vector-valued T-periodic functions that are square integrable over one period, the spectrum of $\mathcal{L}_0$ consists of the discrete set of values of ℓ for which (6) has a nontrivial T-periodic solution. In the rest of the article, we refer to this as the *point spectrum of* $\mathcal{L}_0$. Since we are interested in periodic perturbations, we will primarily be studying the point spectrum. We say that (3) is *spectrally stable* with respect to periodic perturbations if the point spectrum of $\mathcal{L}_0$ does not intersect the open right side of the complex plane. In addition to the point spectrum of $\mathcal{L}_0$, the techniques developed in this article will enable us to find the continuous spectrum.

In order to define the Evans function, we rewrite the ODE system (6) as the following first-order linear system with T-periodic coefficients:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{w} = \mathcal{A}(x, \ell)\,\mathbf{w}, \quad \mathbf{w} = \left(w_1, w_2, w_1', w_2'\right)^{\mathrm{T}}, \tag{8}$$

where $\mathcal{A}(x, \ell)$ is the 4×4 matrix given by

$$\mathcal{A}(x, \ell) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(\alpha + \mathrm{i}\ell + 4U_0^2) & -2U_0^2 & 0 & 0 \\ -2U_0^2 & -(\alpha - \mathrm{i}\ell + 4U_0^2) & 0 & 0 \end{pmatrix}. \tag{9}$$

Let $\Phi(x;\ell)$ be a fundamental matrix for this system, with $\Phi(0;\ell) = I$. Then the Evans function [17] is defined to be

$$M(\kappa, \ell) = \det\left(\Phi(T;\ell) - e^{i\kappa} I\right). \tag{10}$$

Since any solution to the system (8) has the property that $\mathbf{w}(x+T) = \Phi(T;\ell)\mathbf{w}(x)$, there exists a bounded solution to (8) if and only if $M(\kappa,\ell) = 0$ for some $\kappa \in \mathbb{R}$. The zero set of $M(\kappa,\ell)$ for κ real thus corresponds to the continuous spectrum of $\mathcal{L}_0$. However, the point spectrum, which corresponds to T-periodic solutions of (6), is obtained by setting κ to zero. We thus define

$$M_0(\ell) \equiv M(0,\ell). \tag{11}$$

The zero set of $M_0(\ell)$ is the point spectrum of $\mathcal{L}_0$, and the multiplicity of the zero of M_0 is the algebraic multiplicity of each eigenvalue [17]. It is worth mentioning that in the context of Floquet theory (see, e.g., [4], pp. 282–285), the Evans function (10) can be interpreted in the following way: for a given value of ℓ, the values of $e^{i\kappa}$ for which $M(\kappa,\ell) = 0$ correspond to the Floquet multipliers of the system (8).

For all but a few exceptional values of ℓ, the Baker eigenfunctions associated to periodic traveling wave solutions of NLS may be used to construct a fundamental solution matrix for the system (8). (Among the exceptions is the important case $\ell = 0$, which is treated in Section 2.2.) Then the *Floquet discriminant*[1] of the AKNS system (i.e., the Lax pair) may be used to determine the zeros of the Evans function. The following theorem, which establishes the spectral stability of the NLS solution (3) with respect to T-periodic perturbations, is proven in Appendix A.

Theorem 1. (1) *The point spectrum of $\mathcal{L}_0$ defined in (6) consists of a countably infinite number of points on the imaginary axis in the ℓ-plane.*

(2) *Let $\lambda_1 = (\delta/2)(-k' + ik)$, where $k' = \sqrt{1-k^2}$ is the complementary modulus to k. Suppose ℓ is a value for which the equation*

$$(\lambda^2 - \lambda_1^2)(\lambda^2 - \overline{\lambda_1}^2) = \left(\frac{\ell}{4i}\right)^2 \tag{12}$$

has four distinct roots in λ. Then ℓ belongs to the point spectrum if and only if

$$\Delta(\lambda) = 0, 2, \text{ or } -2,$$

where λ is any root of (12), and Δ is the Floquet discriminant of the NLS *solution (3).*

Remark 1. We will refer to the values of ℓ for which (12) has repeated roots as the exceptional values of ℓ. There are four of these and they are purely imaginary. The significance of these values of ℓ is that, when (12) has repeated roots as a polynomial in λ, it is not possible to find four independent solutions of (8) using the squared Baker eigenfunctions. Thus, we cannot use the Floquet discriminant to determine if these values are in the point spectrum, since it is possible for some other nontrivial solution to be periodic.

In addition, in Appendix A.2 we prove the following theorem:

Theorem 2. *The system (8) has a nontrivial solution which is bounded for all $x \in \mathbb{R}$ if and only if ℓ is related by (12) to a point of the continuous Floquet spectrum.*

Thus the continuous spectrum of $\mathcal{L}_0$ consists exactly of the ℓ-values related by (12) to the λ-values in the continuous Floquet spectrum of the Lax pair. The continuous Floquet spectrum of the Lax pair in the complex λ-plane is illustrated, for several specific values of the parameters, in Fig. 7. It consists of the real line and two bands. Under the relation (12), the real line corresponds to the imaginary axis in the complex ℓ-plane minus the interval $i(-4\sqrt{M}, 4\sqrt{M})$ where M is the minimum value for the polynomial of the LHS of (12) for λ real. The two bands correspond to the figure eight curves which intersect the open right side of the complex ℓ-plane centered at the origin (see Fig. 1). This implies that solution (3) is unstable with respect to bounded perturbations, which is consistent with the fact that complex double points in the spectrum of the Lax pair usually give rise to instabilities [13,14,29].

2.2. The NLS Evans function at the origin

For later use in studying the CGL perturbation, we need to characterize the eigenvalue $\ell = 0$ for the problem (6) arising from the linearization of NLS about the T-periodic solution (3). Since the NLS equation (1) has a Hamiltonian structure which admits two symmetries (translation of the variable x and phase change $q_i \to q_i e^{i\theta}$ for any real constant θ), we know that the algebraic multiplicity of the zero eigenvalue will be at least four [18,19]. This is because each of the two symmetries gives rise to an eigenvector and a generalized eigenvector. (The existence of the generalized eigenvector is a consequence of formula (3.3) of [19], in which $\partial_\sigma \phi_\omega$ is the generalized eigenvector corresponding to the eigenvector $T_\sigma \phi_\omega$). We show that the algebraic multiplicity is exactly four by showing that the lowest-order nonzero derivative of the Evans function (11) at $\ell = 0$ is the fourth derivative. More precisely, we have the following theorem:

[1] The Floquet discriminant is defined in Appendix A—see Eq. (81).

Theorem 3. *The Evans function $M_0(\ell)$ defined in (11) is such that*

$$M_0(0) = M_0'(0) = M_0''(0) = M_0'''(0) = 0, \tag{13}$$

where the primes denote derivatives with respect to ℓ. Furthermore, the fourth derivative, which is given by

$$M_0^{(4)}(0) = 384\frac{\left(E^2 + K(K-2E)k'^2\right)^2}{k^4\,k'^4\,\delta^8}, \tag{14}$$

is nonzero for all $k \in (0, 1)$. Thus, the eigenvalue $\ell = 0$ is of algebraic multiplicity exactly four.

Here, $E = E(k)$ is the complete elliptic integral of the second kind.

For proving this theorem, and for later calculations, we will need a basis for the solution space of (8) when $\ell = 0$, i.e., the homogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{U} = \mathcal{A}(x, 0)\,\mathbf{U}. \tag{15}$$

It turns out that (15) can be solved explicitly. To do so, we use the two symmetries of NLS mentioned above, translation in x and phase change. The presence of these two symmetries implies the following T-periodic solutions to (15):

$$\mathbf{V}_1^h = \begin{bmatrix} U_0' \\ U_0' \\ U_0'' \\ U_0'' \end{bmatrix}, \qquad \mathbf{V}_2^h = \begin{bmatrix} U_0 \\ -U_0 \\ U_0' \\ -U_0' \end{bmatrix}. \tag{16}$$

One can then use these two solutions to perform reduction of order on the homogeneous system. The two-dimensional homogeneous linear system for two unknowns obtained that way turns out to be diagonal and thus can be solved by quadrature. The components of a third solution $\mathbf{V}_3^h$ to (15) are given by

$$\begin{aligned}\left(\mathbf{V}_3^h\right)_1 &= -\delta k\left[k'^2\mathrm{dn}(\delta x)\mathrm{sn}(\delta x)\,\delta x + (2k^2-1)\left(\frac{E}{K}\delta x + Z(\delta x)\right)\right]\\ \left(\mathbf{V}_3^h\right)_2 &= \left(\mathbf{V}_3^h\right)_1, \qquad \left(\mathbf{V}_3^h\right)_3 = \left(\mathbf{V}_3^h\right)_4 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathbf{V}_3^h\right)_1.\end{aligned} \tag{17}$$

The components of a fourth solution $\mathbf{V}_4^h$ are given by

$$\begin{aligned}\left(\mathbf{V}_4^h\right)_1 &= -\frac{k}{k'^2}\left[\left(\frac{E}{K} - k'^2\right)\mathrm{cn}(\delta x)\,\delta x + \mathrm{cn}(\delta x)Z(\delta x) - \mathrm{dn}(\delta x)\mathrm{sn}(\delta x)\right]\\ \left(\mathbf{V}_4^h\right)_2 &= -\left(\mathbf{V}_4^h\right)_1, \qquad \left(\mathbf{V}_4^h\right)_3 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathbf{V}_4^h\right)_1, \qquad \left(\mathbf{V}_4^h\right)_4 = -\left(\mathbf{V}_4^h\right)_3.\end{aligned} \tag{18}$$

Note that while $\mathbf{V}_1^h$ and $\mathbf{V}_2^h$ are periodic, no linear combination of $\mathbf{V}_3^h$ and $\mathbf{V}_4^h$ is periodic.

Proof of Theorem 3. Since the algebraic multiplicity of the zero eigenvalue is at least four (due to the Hamiltonian structure) and the multiplicity of the zero of the Evans function equals the algebraic multiplicity of the eigenvalue, Eq. (13) follows immediately. Nevertheless, we proceed to show (13) directly because some of the results that will be obtained in the process will be useful later in the paper.

Let $\mathbf{w}_i(x;\ell)$, $i = 1 \ldots 4$, be four linearly independent solutions of (8) and let $W(x;\ell)$ be a fundamental matrix solution of (8) with columns $\mathbf{w}_i$. In order to compute the derivative of the Evans function at $\ell = 0$, we expand each of the solutions as

$$\mathbf{w}_i(x;\ell) = \mathbf{w}_i^0(x) + \ell\mathbf{w}_i^1(x) + \ell^2\mathbf{w}_i^2(x) + \mathcal{O}(\ell^3). \tag{19}$$

The $\mathbf{w}_i^0(x)$ are solutions of (15), so they are linear combinations of the $\mathbf{V}_i^h$, $i = 1 \ldots 4$ given in (16), (17) and (18). We make the choice

$$\mathbf{w}_i^0(x) = \mathbf{V}_i^h(x).$$

Recursively, the higher-order terms $\mathbf{w}_i^j$ for $j \geq 1$ satisfy the nonhomogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{w}_i^j = \mathcal{A}(x, 0)\,\mathbf{w}_i^j + \mathbf{M}_i^{j-1}, \tag{20}$$

where

$$\mathbf{M}_i^j = \begin{bmatrix} 0 \\ 0 \\ -\mathrm{i}\left(\mathbf{w}_i^j\right)_1 \\ \mathrm{i}\left(\mathbf{w}_i^j\right)_2 \end{bmatrix},$$

and $\left(\mathbf{w}_i^j\right)_k$ denotes the kth component of the vector $\mathbf{w}_i^j$. Since we have a fundamental set of solutions for the homogeneous system, we can solve (20) by variation of parameters method. For the higher-order terms $\mathbf{w}_i^j$ we choose

$$\mathbf{w}_i^j(x) = W_0(x) \int_0^x W_0^{-1}(z)\,\mathbf{M}_i^{j-1}(z)\mathrm{d}z, \quad i = 1\ldots 4,\ j \geq 1,$$

where $W_0(x) \equiv W(x;0)$ is the fundamental matrix solution for the system (15), with columns $\mathbf{V}_i^h$. The formulas for $\mathbf{w}_1^1$ and $\mathbf{w}_2^1$ can be written in a compact way as

$$\mathbf{w}_1^1 = \frac{\mathrm{i}}{2}\left(\begin{bmatrix} -xU_0 \\ xU_0 \\ -(xU_0)' \\ (xU_0)' \end{bmatrix} + \mathbf{V}_4^h\right), \qquad \mathbf{w}_2^1 = \frac{\mathrm{i}}{2\delta^2(1-2k^2)}\left(\begin{bmatrix} (xU_0)' \\ (xU_0)' \\ (xU_0)'' \\ (xU_0)'' \end{bmatrix} + \mathbf{V}_3^h\right), \tag{21}$$

where U_0 is the solution of NLS given in (3).

The expressions for $\mathbf{w}_3^1$, $\mathbf{w}_4^1$, $\mathbf{w}_1^2$, and $\mathbf{w}_2^2$ are rather complicated. However, for the purpose of calculating the derivatives of the Evans functions, we only need their expressions evaluated at $x = T = 4K/\delta$:

$$\mathbf{w}_3^1(T) = \frac{-2\mathrm{i}kE}{k'^2}\begin{bmatrix} \dfrac{4k^2E}{\delta k'^2} \\ -\dfrac{4k^2E}{\delta k'^2} \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{w}_4^1(T) = \frac{2\mathrm{i}kE}{\delta k'^2}\begin{bmatrix} \delta^{-1} \\ \delta^{-1} \\ \dfrac{4k^2E}{k'^2} \\ \dfrac{4k^2E}{k'^2} \end{bmatrix} \tag{22}$$

$$\mathbf{w}_1^2(T) = \frac{-1}{\delta^2kk'^2}\begin{bmatrix} E(2k^2-1)+Kk'^2 \\ E(2k^2-1)+Kk'^2 \\ \dfrac{2\delta\left(k'^4K^2+2k'^2\left(2k^2-1\right)EK-\left(3k^2-4k^4-1\right)E^2\right)}{k'^2} \\ \dfrac{2\delta\left(k'^4K^2+2k'^2\left(2k^2-1\right)EK-\left(3k^2-4k^4-1\right)E^2\right)}{k'^2} \end{bmatrix}, \tag{23}$$

$$\mathbf{w}_2^2(T) = \frac{-1}{\delta^2kk'^2}\begin{bmatrix} -\dfrac{2\left(k'^4K^2-2k'^2EK+\left(k^2+1\right)E^2\right)}{\delta k'^2} \\ \dfrac{2\left(k'^4K^2-2k'^2EK+\left(k^2+1\right)E^2\right)}{\delta k'^2} \\ Kk'^2-E \\ -Kk'^2+E \end{bmatrix}. \tag{24}$$

In order to compute the Evans function (11), we need solutions $\boldsymbol{\phi}_i(x;\ell)$, $i = 1\ldots 4$ of (8) with initial condition $\boldsymbol{\phi}_i(0,\ell) = \mathbf{e}_i$, where $\mathbf{e}_i$ are the standard basis vectors of $\mathbb{R}^4$. The $\boldsymbol{\phi}_i(x;\ell)$ are thus the columns of the matrix $\Phi(x;\ell)$ used in the definition of the Evans function (10). Matching the standard initial conditions with those of the $\mathbf{w}_i$, one finds that

$$\boldsymbol{\phi}_1 = \frac{1}{2\delta k}\left(\mathbf{w}_3+\mathbf{w}_2\right),$$
$$\boldsymbol{\phi}_2 = \frac{1}{2\delta k}\left(\mathbf{w}_3-\mathbf{w}_2\right),$$
$$\boldsymbol{\phi}_3 = \frac{1}{2\delta^3 k}\left(\delta^2\mathbf{w}_4-\mathbf{w}_1\right),$$
$$\boldsymbol{\phi}_4 = \frac{-1}{2\delta^3 k}\left(\delta^2\mathbf{w}_4+\mathbf{w}_1\right).$$

The Evans function (11) can then be expressed as

$$M_0(\ell) = \frac{1}{4\delta^6 k^4} \widetilde{M}_0(\ell), \tag{25}$$

where

$$\widetilde{M}_0(\ell) = \det\left(W(T;\ell) - A\right), \qquad A = \begin{bmatrix} 0 & \delta k & \delta k & 0 \\ 0 & -\delta k & \delta k & 0 \\ -\delta^3 k & 0 & 0 & \delta k \\ \delta^3 k & 0 & 0 & \delta k \end{bmatrix} \tag{26}$$

and $W(x;\ell)$ is the fundamental matrix solution of (8) with columns $\mathbf{w}_i$, $i = 1 \dots 4$. Note that $W(0;\ell) = A$.

Since $\widetilde{M}_0(\ell)$ differs from $M_0(\ell)$ by a factor which is constant in ℓ, the derivatives of M_0 at $\ell = 0$ are constant multiples of those of $\widetilde{M}_0$. The derivatives of $\widetilde{M}_0$ at $\ell = 0$ can be computed using the expansion (19). As $\mathbf{V}_i^h$, $i = 1, 2$ are T-periodic, the first two columns of $W(T;\ell) - A$ are zero at $\ell = 0$. Thus, the expansion of the determinant $\widetilde{M}_0$ in powers of ℓ has no terms of lower order than ℓ^2. Furthermore, the coefficient of ℓ^2 is given by

$$\frac{1}{2}\widetilde{M}_0''(0) = \det\left(\mathbf{w}_1^1(T), \mathbf{w}_2^1(T), \mathbf{V}_3^h(T) - A_3, \mathbf{V}_4^h(T) - A_4\right), \tag{27}$$

where A_i denotes the ith column of A.

The right-hand side of (27) is easily shown to be zero, as follows. Using the expressions for $\mathbf{w}_1^1$ and $\mathbf{V}_4^h$ given in (21) and (18) we find that the vector

$$\mathbf{w}_{g,1} \equiv \mathbf{w}_1^1 + \frac{\mathrm{i}}{2} E\left(k'^2 K - E\right)\mathbf{V}_4^h \tag{28}$$

is a T-periodic function of x. Taking the difference of the values of the vector $\mathbf{w}_{g,1}$ at $x = 0$ and $x = T$ gives a linear combination of the first and fourth columns of the matrix in (27) which must be zero. Similarly, the vector

$$\mathbf{w}_{g,2} \equiv \mathbf{w}_2^1 - \frac{\mathrm{i}}{2\delta^2} E\left(k'^2 K + E(1 - 2k^2)\right)\mathbf{V}_3^h, \tag{29}$$

is periodic and thus a linear combination of the second and third columns of the matrix in (27) is zero. Note that the fact that we are able to find two T-periodic solutions $\mathbf{w}_{g,i}$ of (20) at the first step ($j = 1$) of the recursion is a consequence of the fact mentioned before, that the Hamiltonian structure implies the existence of two generalized eigenvectors.

It is also possible to show that $\widetilde{M}_0'''(0) = 0$, as follows. The idea is to write $\widetilde{M}_0(\ell)$ as

$$\frac{\widetilde{M}_0(\ell)}{\ell^2} = \det\left(\mathbf{w}_1^1(T) + \mathcal{O}(\ell), \mathbf{w}_2^1(T) + \mathcal{O}(\ell), \mathbf{V}_3^h(T) - A_3 + \mathcal{O}(\ell), \mathbf{V}_4^h(T) - A_4 + \mathcal{O}(\ell)\right). \tag{30}$$

Since the ℓ^1 term of the right-hand side of (30) will be a sum of determinants of matrices each involving three of the four columns of the matrix in (27), it must be zero. Thus, we have that the Evans function at least has a fourth order zero at $\ell = 0$.

Then,

$$\begin{aligned} \frac{\widetilde{M}_0(\ell)}{\ell^2} &= \det\Big[\mathbf{w}_1^1(T) + \mathbf{w}_1^2(T)\ell + \mathcal{O}(\ell^2), \mathbf{w}_2^1(T) + \mathbf{w}_2^2(T)\ell + \mathcal{O}(\ell^2), \\ &\quad \mathbf{V}_3^h(T) - A_3 + \mathbf{w}_3^2(T)\ell + \mathcal{O}(\ell^2), \mathbf{V}_4^h(T) - A_4 + \mathbf{w}_4^2(T)\ell + \mathcal{O}(\ell^2)\Big] \\ &= \ell^2 \det\left(\mathbf{w}_1^2(T), \mathbf{w}_2^2(T), \mathbf{V}_3^h(T) - A_3, \mathbf{V}_4^h(T) - A_4\right) + \ell^2 \det\left(\mathbf{w}_1^2(T), \mathbf{w}_2^1(T), \mathbf{w}_3^1(T), \mathbf{V}_4^h(T) - A_4\right) \\ &\quad + \ell^2 \det\left(\mathbf{w}_1^1(T), \mathbf{w}_2^2(T), \mathbf{V}_3^h(T) - A_3, \mathbf{w}_4^1(T)\right) + \ell^2 \det\left(\mathbf{w}_1^1(T), \mathbf{w}_2^1(T), \mathbf{w}_3^1(T), \mathbf{w}_4^1(T)\right) + \mathcal{O}(\ell^3). \end{aligned} \tag{31}$$

It is then straightforward to use (31), and the expressions for $\mathbf{V}_i^h$, $\mathbf{w}_i^1$ and $\mathbf{w}_i^2$ in (17), (18) and (21) through (24), to find

$$\widetilde{M}_0^{(4)}(0) = 1536 \frac{\left(E^2 + K(K - 2E)k'^2\right)^2}{\delta^2 k'^4}.$$

Then one obtains (14) using the relation (25). Using the inequalities $K > E > k'K$, which hold for all moduli $k \in (0, 1)$, we find that

$$E^2 + K(K - 2E)k'^2 > 2k'^2 K(K - E) > 0,$$

and thus the fourth derivative of the Evans function is nonzero for all such k. □

Before the proof of Theorem 3, we established that for $\ell = 0$, the system (8) only has a two-dimensional space of T-periodic solutions, generated by $\mathbf{V}_1^h$ and $\mathbf{V}_2^h$. The eigenvalue problem (6) thus admits a two-dimensional space of eigenvectors for $\ell = 0$. However, since the zero of the Evans function is of multiplicity four, the eigenvalue $\ell = 0$ is actually of algebraic multiplicity four. We now describe two eigenvectors and two generalized eigenvectors corresponding to $\ell = 0$.

We denote the two eigenvectors $\boldsymbol{\omega}_0^1, \boldsymbol{\omega}_0^2 \in \operatorname{Ker}(\mathcal{L}_0)$. They are given by

$$\boldsymbol{\omega}_0^1 = \begin{bmatrix} U_0' \\ U_0' \end{bmatrix}, \qquad \boldsymbol{\omega}_0^2 = \begin{bmatrix} U_0 \\ -U_0 \end{bmatrix}, \tag{32}$$

where U_0 is the NLS solution (3). Again, these solutions of linearized NLS arise from applying the symmetries mentioned above: respectively, translation in x and phase change $q_i \to q_i e^{i\theta}$. As mentioned before, because NLS is a Hamiltonian equation, there are also two generalized eigenvectors $\boldsymbol{\omega}_0^{g,1}, \boldsymbol{\omega}_0^{g,2}$ satisfying

$$\mathcal{L}_0 \boldsymbol{\omega}_0^{g,i} = \boldsymbol{\omega}_0^i, \quad i = 1, 2. \tag{33}$$

The two components of the generalized eigenvector $\boldsymbol{\omega}_0^{g,1}$ (resp. $\boldsymbol{\omega}_0^{g,2}$) are the first two components of $\mathbf{w}_{g,1}$ (resp. $\mathbf{w}_{g,2}$) given in (28) (resp. (29)).

3. The perturbed setting

3.1. Periodic traveling wave solutions to CGL

The CGL perturbation of the NLS system has the following form:

$$\begin{aligned} &i q_{1t} + q_{1xx} + 2q_1^2 q_2 = \epsilon[i r q_1 + i q_{1xx} - 2 i s q_1^2 q_2], \\ &-i q_{2t} + q_{2xx} + 2 q_1 q_2^2 = \epsilon[-i r q_2 - i q_{2xx} + 2 i s q_1 q_2^2]. \end{aligned} \tag{34}$$

Here $\epsilon > 0$ and r, s are real parameters. The system (34) bears the same relationship to the focusing CGL equation as (1) does to focusing NLS: if solution (q_1, q_2) satisfies the reality condition $q_2 = \overline{q_1}$ then q_1 solves focusing CGL. Note that the CGL also shares the NLS symmetries of translation in x and phase change.

We again make the change of variables $q_1 = e^{-i\alpha t} u_1$, $q_2 = e^{i\alpha t} u_2$, and obtain the following system:

$$\begin{aligned} &i u_{1t} + \alpha u_1 + u_{1xx} + 2u_1^2 u_2 = \epsilon[i r u_1 + i u_{1xx} - 2 i s u_1^2 u_2] \\ &-i u_{2t} + \alpha u_2 + u_{2xx} + 2 u_1 u_2^2 = \epsilon[-i r u_2 - i u_{2xx} + 2 i s u_1 u_2^2], \end{aligned} \tag{35}$$

with the reality condition $u_2 = \overline{u_1}$.

The solution to NLS given in (3) persists as a T-periodic solution of the CGL (35) if r and s satisfy a certain algebraic condition. More precisely, it was shown in [11] (see Prop. 5.0.1 in that paper) that the solution (3) for NLS persists as a solution of the CGL (35) that is stationary and T-periodic in x if and only if r is sufficiently large, i.e.,

$$r \geq \frac{\delta^2 \pi^2}{4K^2}, \tag{36}$$

and the modulus k is uniquely determined by the equation[2]

$$(E - K(k')^2) r = \frac{\delta^2}{3} \left[(4s+1)((2k^2-1)E + (k')^2 K) - 6 s k^2 (k')^2 K \right]. \tag{37}$$

(This equation is equivalent to Eq. (36) in the proof of Prop. 5.0.1 in [11], evaluated in terms of standard elliptic integrals. Note that T is taken to be equal to 1 in that proof. Note also that our cnoidal solutions correspond to setting the integral parameter $m = 2$ in [11], and our parameter s corresponds to the parameter q in [11].) This solution to (35) satisfies $u_1(x,t) = U(x)$, $u_2(x,t) = \overline{U(x)}$, and we expand $U(x)$ in ϵ as

$$U(x) = U_0(x) + \epsilon U_1(x) + \mathcal{O}(\epsilon^2), \tag{38}$$

where U_0 is the solution of NLS given in (3).

[2] For a given s-value and modulus k, the r-value given by (37) does not automatically satisfy the inequality (36); see Fig. 2 for a plot of the region in k–s plane where the inequality holds.

In order to investigate stability properties of the CGL solution U, we will need the expression giving U_1. To find it, we substitute $u_1 = U$ and $u_2 = \overline{U}$ into (35). At first order in ϵ, one finds that the vector $\mathbf{U} = \left(U_1, \overline{U_1}, U_1', \overline{U_1}'\right)^{\mathrm{T}}$ satisfies a nonhomogeneous linear dynamical system. This system is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{U} = \mathcal{A}(x, 0)\mathbf{U} + \mathbf{N}, \tag{39}$$

where $\mathcal{A}(x, 0)$ is obtained by setting $\ell = 0$ in the matrix given in (9), and the nonhomogeneous term $\mathbf{N}$ is given by

$$\mathbf{N} = \begin{bmatrix} 0 \\ 0 \\ \mathrm{i}\,r\,U_0 + \mathrm{i}\,U_0'' - 2\mathrm{i}\,s\,U_0^3 \\ -\mathrm{i}\,r\,U_0 - \mathrm{i}\,U_0'' + 2\mathrm{i}\,s\,U_0^3 \end{bmatrix}.$$

Recall from Section 2.2 that the corresponding homogeneous system (15) has only a two-dimensional space of T-periodic solutions generated by $\mathbf{V}_1^h$ and $\mathbf{V}_2^h$. Given the four linearly independent solutions to (15) which we know, one can use variation of parameters to find a particular solution $\mathbf{U}$ to (39). Requiring this solution to be T-periodic leads to the algebraic relation (37) between the constants k, r, δ, and s. This last condition was also obtained in [11], but we also explicitly calculate the first component of $\mathbf{U}$, given by

$$U_1 = \mathrm{i}\delta k(s+1)\mathrm{cn}(\delta x)\left[-\frac{2}{3}\ln\left(\theta_4\left(\frac{\pi\delta x}{2K}\right)\right) + \frac{K\left(2\mathrm{dn}(\delta x)\mathrm{sn}(\delta x)Z(\delta x) - Z^2(\delta x)\mathrm{cn}(\delta x) + \mathrm{cn}^3(\delta x)k^2\right)}{3\mathrm{cn}(\delta x)(E + (k^2-1)K)}\right], \tag{40}$$

where Z is the Jacobi zeta function and θ_4 is a Jacobi theta function. (For definitions and conventions for these and other relatives of the elliptic functions, we refer the reader to [6].)

Note that any linear combination of $\mathbf{V}_1^h$ or $\mathbf{V}_2^h$ can be added to $\mathbf{U}$ without losing the periodicity property. This last fact can be explained by the presence of the two symmetries of CGL mentioned above: adding a multiple of the top component of $\mathbf{V}_1^h$ corresponds to adding translation in x to the CGL perturbation, while adding a multiple of the top component of $\mathbf{V}_2^h$ corresponds to adding a phase change.

Now that we know more about the dependence of $U(x)$ in ϵ, we can study its spectral stability as a solution of CGL.

3.2. Linearized CGL

We linearize the system of equations (35) about the stationary T-periodic solution $u_1(x,t) = U(x)$, $u_2(x,t) = \overline{U(x)}$, where U is given in (38). We do this in a similar way as it is done for NLS in Section 2.1. Setting $u_1 = U + v_1$, $u_2 = \overline{U} + v_2$ and discarding all higher-order terms in the v_i's, we obtain

$$\begin{aligned} &\mathrm{i}v_{1t} + \alpha v_1 + v_{1xx} + 4|U|^2 v_1 + 2U^2 v_2 = \epsilon[\mathrm{i}rv_1 + \mathrm{i}v_{1xx} - 4\mathrm{i}s|U|^2 v_1 - 2\mathrm{i}sU^2 v_2], \\ &-\mathrm{i}v_{2t} + \alpha v_2 + v_{1xx} + 4|U|^2 v_2 + 2\overline{U}^2 v_1 = \epsilon[-\mathrm{i}rv_2 - \mathrm{i}v_{2xx} + 4\mathrm{i}s|U|^2 v_2 + 2\mathrm{i}s\overline{U}^2 v_1]. \end{aligned} \tag{41}$$

Next, by analogy with Section 2 we substitute into (41) the ansatz

$$v_1(x,t) = \mathrm{e}^{\ell t} w_1(x), \qquad v_2(x,t) = \mathrm{e}^{\ell t} w_2(x).$$

This results in the following ODE system for w_1, w_2

$$\begin{aligned} &\mathrm{i}\ell w_1 + \alpha w_1 + w_{1xx} + 4|U|^2 w_1 + 2U^2 w_2 = \epsilon[\mathrm{i}rw_1 + \mathrm{i}w_{1xx} - 4\mathrm{i}s|U|^2 w_1 - 2\mathrm{i}sU^2 w_2] \\ &-\mathrm{i}\ell w_2 + \alpha w_2 + w_{1xx} + 4|U|^2 w_2 + 2\overline{U}^2 w_1 = \epsilon[-\mathrm{i}rw_2 - \mathrm{i}w_{2xx} + 4\mathrm{i}s|U|^2 w_2 + 2\mathrm{i}s\overline{U}^2 w_1], \end{aligned}$$

which can be rewritten as the eigenvalue problem

$$\mathcal{L}\boldsymbol{\omega} = \ell\boldsymbol{\omega}, \quad \boldsymbol{\omega} \equiv (w_1, w_2)^{\mathrm{T}}. \tag{42}$$

If we expand $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \mathcal{O}(\epsilon^3)$, then $\mathcal{L}_0$ is the operator arising from the linearization of NLS given in (7) and $\mathcal{L}_1$ is given by

$$\mathcal{L}_1 = I_2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \begin{pmatrix} r - 4sU_0^2 & 2U_0\,(2\mathrm{i}U_1 - sU_0) \\ 2U_0\,(2\mathrm{i}U_1 - sU_0) & r - 4sU_0^2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since we are interested in periodic perturbations, we restrict the eigenvalue problem (42) to the space $(L_2(T))^2$ of T-periodic square integrable functions.

3.3. Eigenvalues near the origin

In this section, we study stability properties of $U(x)$ as a solution to CGL by studying the spectrum of the operator $\mathcal{L}$. We use a perturbative argument to study the eigenvalues of $\mathcal{L}$ near $\ell = 0$.

The linearization of CGL about the T-periodic solution (38) gives rise to the eigenvalue problem (42) defined over the space $(L_2(T))^2$ of T-periodic square integrable vector-valued functions. Two eigenvectors $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \in \mathrm{Ker}(\mathcal{L})$ are given by

$$\boldsymbol{\omega}^1 = \begin{bmatrix} U' \\ U' \end{bmatrix}, \qquad \boldsymbol{\omega}^2 = \begin{bmatrix} U \\ -U \end{bmatrix}, \tag{43}$$

where U is the solution of CGL given in (38). These arise, as in (32), from applying the CGL symmetries of translation in x and phase change, respectively. The eigenvectors (43) can be expanded in ϵ as

$$\begin{aligned} \boldsymbol{\omega}^1 &= \boldsymbol{\omega}_0^1 + \epsilon \boldsymbol{\omega}_1^1 + \epsilon^2 \boldsymbol{\omega}_2^1 + \mathcal{O}(\epsilon^3), \\ \boldsymbol{\omega}^2 &= \boldsymbol{\omega}_0^2 + \epsilon \boldsymbol{\omega}_1^2 + \epsilon^2 \boldsymbol{\omega}_2^2 + \mathcal{O}(\epsilon^3), \end{aligned} \tag{44}$$

where $\boldsymbol{\omega}_0^i$ are the two vector-valued functions of $\mathrm{Ker}(\mathcal{L}_0)$ given in (32), and $\boldsymbol{\omega}_1^i$ are given by

$$\boldsymbol{\omega}_1^1 = \begin{bmatrix} U_1' \\ U_1' \end{bmatrix}, \qquad \boldsymbol{\omega}_1^2 = \begin{bmatrix} U_1 \\ -U_1 \end{bmatrix}, \tag{45}$$

where U_1 is the coefficient of ϵ in the solution of CGL and is given in (40). Furthermore, the vector-valued functions $\boldsymbol{\omega}_0^i$, $\boldsymbol{\omega}_1^i$, and $\boldsymbol{\omega}_2^i$ in (44) satisfy the differential equations

$$\begin{aligned} &\mathcal{L}_0 \boldsymbol{\omega}_1^i + \mathcal{L}_1 \boldsymbol{\omega}_0^i = 0, \\ &\mathcal{L}_1 \boldsymbol{\omega}_1^i + \mathcal{L}_2 \boldsymbol{\omega}_0^i + \mathcal{L}_0 \boldsymbol{\omega}_2^i = 0, \quad i = 1, 2. \end{aligned} \tag{46}$$

These equations are obtained by inserting the right-hand sides of (44) into (42) with $\ell = 0$ and then writing the relations occurring at the first and second order in ϵ.

When ϵ is not zero, the NLS is perturbed to the CGL (35) which does not admit a Hamiltonian structure. As a consequence, it is no longer automatic that the algebraic multiplicity of the eigenvalue $\ell = 0$ is higher than two, and two eigenvalues may move away from the origin. In Appendix B we show that this does occur, and moreover the emerging eigenvalues are distinct and analytic functions of ϵ. (Note that the facts proven in Appendix B together with the formulas (51) derived below can be obtained directly as a consequence of the theory presented in Section 4 of [24]. Here and in Appendix B, we present an alternative derivation of these results for our particular case.)

In order to track the emerging eigenvalues, we will use an argument based on the Fredholm alternative. We begin by supposing that $\boldsymbol{\omega}_\epsilon$ is an eigenvector corresponding to a nonzero eigenvalue ℓ_ϵ, with both $\boldsymbol{\omega}_\epsilon$ and ℓ_ϵ analytic in ϵ. We can thus expand $\boldsymbol{\omega}_\epsilon$ and ℓ_ϵ in the following way:

$$\begin{aligned} \ell_\epsilon &= \epsilon \ell_1 + \epsilon^2 \ell_2 + \mathcal{O}(\epsilon^3), \\ \boldsymbol{\omega}_\epsilon &= \boldsymbol{\omega}_0 + \epsilon \boldsymbol{\omega}_1 + \epsilon^2 \boldsymbol{\omega}_2 + \mathcal{O}(\epsilon^3). \end{aligned} \tag{47}$$

At first, second, and third order in ϵ, we find the following equations for $\boldsymbol{\omega}_0$, $\boldsymbol{\omega}_1$, and $\boldsymbol{\omega}_2$:

$$\mathcal{L}_0 \boldsymbol{\omega}_0 = 0, \tag{48}$$

$$\mathcal{L}_0 \boldsymbol{\omega}_1 = \ell_1 \boldsymbol{\omega}_0 - \mathcal{L}_1 \boldsymbol{\omega}_0, \tag{49}$$

$$\mathcal{L}_0 \boldsymbol{\omega}_2 = \ell_1 \boldsymbol{\omega}_1 + \ell_2 \boldsymbol{\omega}_0 - \mathcal{L}_1 \boldsymbol{\omega}_1 - \mathcal{L}_2 \boldsymbol{\omega}_0. \tag{50}$$

We will not solve these equations but rather find a solvability condition that guarantees nontrivial T-periodic solutions.

Lemma 1. *The system (48)–(50) admits a nontrivial T-periodic solution $(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ only if $\ell_1 = 0$, or if ℓ_1 is given by either of the expressions*

$$\ell_1 = \frac{\left\langle \mathcal{L}_1 \boldsymbol{\omega}_0^{g,1}, J\boldsymbol{\omega}_0^1 \right\rangle - \left\langle \boldsymbol{\omega}_1^1, J\boldsymbol{\omega}_0^1 \right\rangle}{\left\langle \boldsymbol{\omega}_0^{g,1}, J\boldsymbol{\omega}_0^1 \right\rangle} \quad \text{or} \quad \ell_1 = \frac{\left\langle \mathcal{L}_1 \boldsymbol{\omega}_0^{g,2}, J\boldsymbol{\omega}_0^2 \right\rangle - \left\langle \boldsymbol{\omega}_2^1, J\boldsymbol{\omega}_0^2 \right\rangle}{\left\langle \boldsymbol{\omega}_0^{g,2}, J\boldsymbol{\omega}_0^2 \right\rangle}. \tag{51}$$

Proof. The vector $\boldsymbol{\omega}_0$ is in the kernel of $\mathcal{L}_0$. Thus $\boldsymbol{\omega}_0$ will be some linear combination of $\boldsymbol{\omega}_0^1$ and $\boldsymbol{\omega}_0^2$:

$$\boldsymbol{\omega}_0 = \alpha \boldsymbol{\omega}_0^1 + \beta \boldsymbol{\omega}_0^2, \tag{52}$$

where α and β are complex constants. Then, using (33), the first equation in (46) and linear superposition, we obtain the following solution to (49)

$$\boldsymbol{\omega}_1 = \alpha\boldsymbol{\omega}_1^1 + \beta\boldsymbol{\omega}_1^2 + \ell_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right), \tag{53}$$

where α and β are the constants in (52), $\boldsymbol{\omega}_1^i$ are given in (45), and $\boldsymbol{\omega}_0^{g,i}$ are the generalized eigenvectors of $\mathcal{L}_0$ satisfying the relations (33). We can eliminate the $\mathcal{L}_2$ term from (50). Indeed, if we use the second equation in (46) and the expression for $\boldsymbol{\omega}_1$ given in (53), one finds that

$$\mathcal{L}_1\boldsymbol{\omega}_1 + \mathcal{L}_2\boldsymbol{\omega}_0 = \ell_1\mathcal{L}_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right) - \mathcal{L}_0\left(\alpha\boldsymbol{\omega}_2^1 + \beta\boldsymbol{\omega}_2^2\right), \tag{54}$$

and (50) then becomes

$$\mathcal{L}_0\boldsymbol{\omega}_2 = \ell_1\boldsymbol{\omega}_1 + \ell_2\boldsymbol{\omega}_0 - \ell_1\mathcal{L}_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right) + \mathcal{L}_0\left(\alpha\boldsymbol{\omega}_2^1 + \beta\boldsymbol{\omega}_2^2\right). \tag{55}$$

To find a solvability condition that guarantees T-periodic solutions to (55), we first define the usual inner product

$$\langle \mathbf{X}_1, \mathbf{X}_2\rangle \equiv \int_0^{\mathrm{T}} \mathbf{X}_1^t(x)\overline{\mathbf{X}}_2(x)\mathrm{d}x, \quad \mathbf{X}_i \in (L_2(T))^2. \tag{56}$$

With respect to (56), the adjoint of $\mathcal{L}_0$ from (7) is given by

$$\mathcal{L}_0^\dagger = -\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \begin{pmatrix} 4U_0^2+\alpha & 2U_0^2 \\ 2U_0^2 & 4U_0^2+\alpha\end{pmatrix}\right]J, \quad J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i}\end{pmatrix}. \tag{57}$$

It is then easy to verify that the kernel of $\mathcal{L}_0^\dagger$ is generated by $J\boldsymbol{\omega}_0^i$, $i = 1, 2$, where $\boldsymbol{\omega}_0^i$ are the generators of the kernel of $\mathcal{L}_0$.

We can now obtain compatibility conditions for (55) by taking the scalar product with elements of the kernel of $\mathcal{L}_0^\dagger$:

$$\begin{aligned}\left\langle \mathcal{L}_0\boldsymbol{\omega}_2, J\boldsymbol{\omega}_0^i\right\rangle &= \ell_1\left\langle\boldsymbol{\omega}_1, J\boldsymbol{\omega}_0^i\right\rangle + \ell_2\left\langle\boldsymbol{\omega}_0, J\boldsymbol{\omega}_0^i\right\rangle - \ell_1\left\langle\mathcal{L}_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right), J\boldsymbol{\omega}_0^i\right\rangle \\ &\quad + \left\langle\mathcal{L}_0\left(\alpha\boldsymbol{\omega}_2^1 + \beta\boldsymbol{\omega}_2^2\right), J\boldsymbol{\omega}_0^i\right\rangle, \quad i = 1, 2.\end{aligned} \tag{58}$$

As $J\boldsymbol{\omega}_0^i$ is in the kernel of $\mathcal{L}_0^\dagger$, the left-hand side and the last term of the right-hand side of (58) are zero. Taking this into account, we can rewrite (58) using the expression for $\boldsymbol{\omega}_1$ given in (53)

$$\begin{aligned}0 &= \ell_1^2\left\langle\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}, J\boldsymbol{\omega}_0^i\right\rangle + \ell_1\left(\left\langle\alpha\boldsymbol{\omega}_1^1 + \beta\boldsymbol{\omega}_1^2, J\boldsymbol{\omega}_0^i\right\rangle - \left\langle\mathcal{L}_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right), J\boldsymbol{\omega}_0^i\right\rangle\right) \\ &\quad + \ell_2\left\langle\alpha\boldsymbol{\omega}_0^1 + \beta\boldsymbol{\omega}_0^2, J\boldsymbol{\omega}_0^i\right\rangle, \quad i = 1, 2.\end{aligned} \tag{59}$$

This is a homogeneous linear system of equations for α and β and we are looking for the values of ℓ_i for which (59) has a nontrivial solution. The system can be simplified further by making the observation that the components of the vectors $\boldsymbol{\omega}_0^i$, $\boldsymbol{\omega}_1^i$, and $\boldsymbol{\omega}_0^{g,i}$ are even functions of x with respect to the axis $x = T/2$ for $i = 1$ and odd for $i = 2$. Thus, any inner product in (59) involving two different superscripts will be zero. Furthermore, because J is skew-adjoint, the last term of (59) is always zero. We reduce to the two equations

$$\begin{aligned}0 &= \alpha\ell_1\left(\ell_1\left\langle\boldsymbol{\omega}_0^{g,1}, J\boldsymbol{\omega}_0^1\right\rangle + \left\langle\boldsymbol{\omega}_1^1, J\boldsymbol{\omega}_0^1\right\rangle - \left\langle\mathcal{L}_1\boldsymbol{\omega}_0^{g,1}, J\boldsymbol{\omega}_0^1\right\rangle\right), \\ 0 &= \beta\ell_1\left(\ell_1\left\langle\boldsymbol{\omega}_0^{g,2}, J\boldsymbol{\omega}_0^2\right\rangle + \left\langle\boldsymbol{\omega}_1^2, J\boldsymbol{\omega}_0^2\right\rangle - \left\langle\mathcal{L}_1\boldsymbol{\omega}_0^{g,2}, J\boldsymbol{\omega}_0^2\right\rangle\right).\end{aligned} \tag{60}$$

The system (60) has a nontrivial solution (α, β) if $\ell_1 = 0$, which corresponds to the generators of the kernel of $\mathcal{L}$ given in (43), or if ℓ_1 is given by either of the expressions in (51). □

The components of the vectors $\boldsymbol{\omega}_0^i$ and $\boldsymbol{\omega}_1^i$ are real and those of $\boldsymbol{\omega}_0^{g,i}$ are purely imaginary making the expressions in (51) all real. Thus (42) has an eigenvalue on the right side of the complex plane for small positive values of ϵ if one of the two solutions in (51) is positive. This gives an instability condition for the solution (38).

Note that in what follows, we make the simplifying assumption that $\delta = 1$. This is done without loss of generality because CGL in the form (34) has the following scaling symmetry: if $q_2(x,t) = \overline{q_1(x,t)} = f(x,t)$ is a solution for a given value of the coefficient $r = r_0$, then $q_2(x,t) = \overline{q_1(x,t)} = \gamma f(\gamma x, \gamma^2 t)$ is a solution corresponding to the value of the coefficient $r = r_0/\gamma^2$. This symmetry can thus be used to set δ to 1.

The condition that the first expression in (51) be positive is equivalent to the inequality

$$\begin{aligned}&4k'^2((214k^4-223k^2-15)s+4k^4+197k^2-225)K^4-420(1+s)E^4\\&\quad+8((214k^6-657k^4+428k^2+39)s+4k^6+78k^4-412k^2+354)EK^3\\&\quad-168(4k^4+16sk^2k'^2+4s+19k'^2)E^2K^2+840(s+2k'^2)E^3K\\&\quad-210\,(1+s)\,K((1+k^2)E^2+(1+7k^2+k^4)K^2-2(1+4k^2+k^4)EK)I_1\\&\quad-315k^4\,(1+s)\,K^2(E-5K-6k^2K)I_2-1575k^6\,(1+s)\,K^3I_3>0,\end{aligned}\tag{61}$$

where

$$I_1\equiv\int_0^{4K}Z^2(x)\mathrm{d}x,\qquad I_2\equiv\int_0^{4K}Z^2(x)\mathrm{sn}^4(x)\mathrm{d}x,\qquad I_3\equiv\int_0^{4K}Z^2(x)\mathrm{sn}^6(x)\mathrm{d}x.$$

Furthermore, The condition that the second expression in (51) be positive is equivalent to the inequality

$$\begin{aligned}&4k'^2((598k^4-1373k^2+799)s-32k^4-113k^2+169)K^4-420(1+s)E^4\\&\quad 8((598k^6-2108k^4+2624k^2-1138)s-32k^6+97k^4+104k^2-193)EK^3\\&\quad+56((173k^4-308k^2+143)s+7k^2k'^2+8)E^2K^2+840(6sk^2-2s+1)E^3K\\&\quad+210\,(1+s)\,K((1-8k^2-2k^4)K^2+(4k^4+7k^2-2)EK+(1+k^2)E^2)I_1\\&\quad+315\,(1+s)\,k^4K^2(8k^2K+E+3K)I_2-1575\,(1+s)\,k^6K^3I_3>0.\end{aligned}\tag{62}$$

Theorem 4. *When either condition (61) or (62) is met, one eigenvalue emerges from the origin on the right side of the complex ℓ-plane for small positive ϵ.*

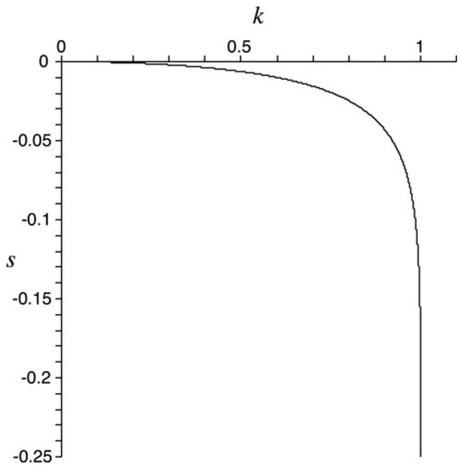

Fig. 2. The region above this curve is where the inequality (36) holds, which is necessary for the persistence of cnoidal waves in the CGL perturbation. (We have used (37) to solve for r in terms of k, s and δ, and used freedom of scaling to set $\delta = 1$.) Along the curve, $s = 0$ when $k = 0$, and $s = -1/4$ when $k = 1$.

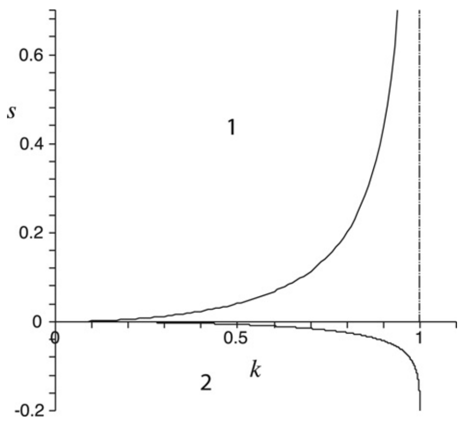

Fig. 3. In region 1, the inequality (61) holds, while region 2 is where the inequality (62) holds.

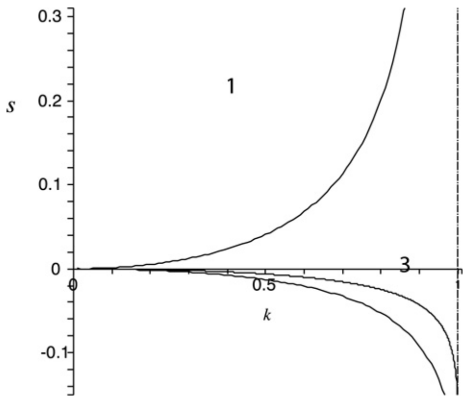

Fig. 4. This shows the curve from Fig. 2 with the curves from Fig. 3. We have verified numerically that the curve from Fig. 2 is always above the bottom curve from Fig. 3, except at $k = 0$ where both have value $s = 0$.

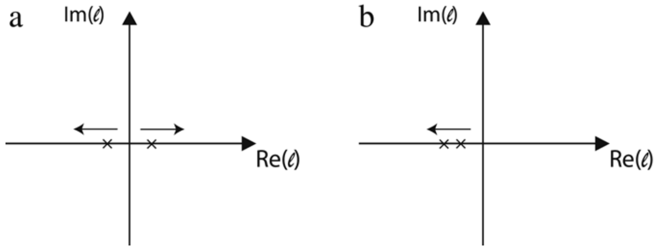

Fig. 5. The behavior of the two emerging eigenvalues, shown in diagram (a) for (k, s)-values in region 1 of Fig. 4, and diagram (b) for values in region 3 of Fig. 4.

In other words, conditions (61) and (62) are spectral instability criteria for the solution of CGL given in (38). Figs. 2 through 4 show the regions in the k–s plane in which the inequalities for existence (36) and instability (61) and (62) are satisfied. Depending on the values of k and s, there are two possible types of behaviors for the two emerging eigenvalues (see Fig. 5). In the first case, the two eigenvalues move to the left on the real line making our analysis inconclusive since other eigenvalues could possibly emerge from zeros of the Evans function elsewhere on the imaginary axis. In the second case, the eigenvalues move in opposite directions on the real line causing the solution (38) to be spectrally unstable.

4. Discussion

In this article, we have obtained both the continuous and point spectrum for the linearized NLS about cnoidal waves (see Theorems 1 and 2) and shown that these solutions are spectrally stable with respect to perturbations that have the same periodicity. Furthermore, in Theorem 4, we obtain the conditions under which eigenvalues of the linearized CGL bifurcate from the origin to the right side of the complex plane as NLS is perturbed to CGL.

It is worth noting that the situation for periodic solutions is different from the case of solitary waves. In the latter case, one is looking for bifurcation from points in the continuous spectrum which, in the case of the NLS, can only occur where the Evans function has branch points [21,23]. In the periodic case, no bifurcation from the continuous spectrum is possible and furthermore it is clear by its definition that the Evans function is analytic everywhere.

Several interesting questions still need to be investigated. One concerns the fact that Theorem 4 only provides a sufficient condition for the instability of the periodic solution of CGL. This is because only bifurcations from the origin are considered. While Theorem 1 identifies a countably infinite number of eigenvalues of the linearized NLS on the imaginary axis, the fate of these eigenvalues when NLS is perturbed to CGL is not known. In particular, one can imagine that eigenvalues bifurcate from the imaginary axis to the right side of the complex plane for small positive values of ϵ, thus creating additional instabilities. In principle, since the squared Baker eigenfunctions provide explicit expressions for the eigenvectors, it is possible to perform a computation based on the Fredholm alternative, such as the one presented in Section 3.3, to determine the fate the eigenvalues located on the imaginary axis. In practice however, the complicated expressions for the eigenvectors given by ratios of theta functions and the fact that the location of the eigenvalues is only given implicitly in Theorem 1 make the computations very difficult to perform. We thus leave such as study to the future.

As is well known, both the focusing and defocusing NLS admit other families of periodic traveling wave solutions. We envision no special difficulty in applying our method to other families of such solutions in both cases. It is important to note however that, in the defocusing case, the continuous Floquet spectrum of the Lax pair consists of gaps and bands on the real axis [2]. This is different from the focusing case, where in addition to the entire real line the continuous spectrum may have additional bands extending above and below the real axis. For cnoidal waves there are 2 bands (shown in Fig. 7) which correspond, via the relation (12), to the figure-eight curve in Fig. 7. In the defocusing case, the absence of these bands should make the computations easier. Moreover, since in the focusing case these band create instabilities with respect to bounded perturbations, we speculate that in the defocusing case the periodic solutions are spectrally stable with respect to such perturbations.

Acknowledgments

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Appendix A. The Evans function for Cnoidal NLS solutions

In this appendix we use solutions of the AKNS spectral problem for focusing NLS solutions of the form

$$q_0(x,t) = \delta k \mathrm{e}^{-\mathrm{i}\alpha t}\mathrm{cn}(\delta x; k), \quad \alpha = \delta^2(1-2k^2) \tag{63}$$

(which corresponds to (3)) to construct solutions of the ODE system (8) using squared AKNS eigenfunctions. We demonstrate that for all but five values of ℓ along the imaginary axis (including $\ell = 0$), such solutions form a basis for the solution space of (8). Using this information, we determine all zeros of the Evans function (other than possibly the four nonzero imaginary values) using the Floquet spectrum of q_0.

A.1. Squared eigenfunctions and the Evans ansatz

In this subsection, we will assume that q is a genus one finite-gap solution of focusing NLS (1). These solutions have the form

$$q_1(x,t) = \overline{q_2(x,t)} = q(x,t) = \mathrm{e}^{-\mathrm{i}\alpha t}U_0(\xi), \quad \xi = x - ct,$$

for real constants α, c and a T-periodic function U_0 which is not necessarily real. (In Section A.2 we will specialize to the case of the cnoidal solution given by (63).) The change of variables $u_1(\xi,t) = \mathrm{e}^{\mathrm{i}\alpha t}q_1(x,t)$, $u_2(\xi,t) = \mathrm{e}^{-\mathrm{i}\alpha t}q_2(x,t)$ will allow us to linearize about a stationary periodic solution when it is applied to system (1), yielding

$$\begin{aligned} &\mathrm{i}u_{1t} + u_{1\xi\xi} - \mathrm{i}cu_{1\xi} + \alpha u_1 + 2u_1^2u_2 = 0,\\ &-\mathrm{i}u_{2t} + u_{2\xi\xi} + \mathrm{i}cu_{2\xi} + \alpha u_2 + 2u_1u_2^2 = 0. \end{aligned} \tag{64}$$

(Note that, here, the time derivative holds ξ fixed.) We linearize this system about the complex-valued solution $u_1 = \overline{u_2} = U_0(\xi)$, yielding

$$\begin{aligned} &\mathrm{i}v_{1t} + v_{1\xi\xi} - \mathrm{i}cv_{1\xi} + \alpha v_1 + 4|U_0|^2v_1 + 2U_0^2v_2 = 0,\\ &-\mathrm{i}v_{2t} + v_{2\xi\xi} + \mathrm{i}cv_{2\xi} + \alpha v_2 + 4|U_0|^2v_2 + 2\overline{U_0}^2v_1 = 0. \end{aligned} \tag{65}$$

We will now show how to obtain solutions for this linearized system, satisfying the ansatz

$$v_1 = \mathrm{e}^{\ell t}w_1(\xi), \qquad v_2 = \mathrm{e}^{\ell t}w_2(\xi). \tag{66}$$

Recall the AKNS system for focusing NLS [1]:

$$\boldsymbol{\psi}_x = \begin{bmatrix} -\mathrm{i}\lambda & \mathrm{i}q \\ \mathrm{i}\overline{q} & \mathrm{i}\lambda \end{bmatrix}\boldsymbol{\psi}, \qquad \boldsymbol{\psi}_t = \begin{bmatrix} \mathrm{i}(|q|^2 - 2\lambda^2) & 2\mathrm{i}\lambda q - q_x \\ 2\mathrm{i}\lambda\overline{q} + \overline{q_x} & \mathrm{i}(2\lambda^2 - |q|^2) \end{bmatrix}\boldsymbol{\psi}. \tag{67}$$

As is well known [14,30], the squares of the components of $\boldsymbol{\psi}$ give a solution of the linearization of (1) at q. This linearization is

$$\begin{aligned} &\mathrm{i}g_t + g_{xx} + 4|q|^2g + 2q^2h = 0,\\ &-\mathrm{i}h_t + h_{xx} + 4|q|^2h + 2\overline{q}^2g = 0, \end{aligned} \tag{68}$$

and the solution is given by $g = \psi_1^2$, $h = \psi_2^2$. It is easy to check that the substitutions

$$q(x,t) = \mathrm{e}^{-\mathrm{i}\alpha t}U_0(\xi), \qquad g(x,t) = \mathrm{e}^{-\mathrm{i}\alpha t}v_1(\xi,t), \qquad h(x,t) = \mathrm{e}^{\mathrm{i}\alpha t}v_2(\xi,t) \tag{69}$$

turn (68) into (65).

It is also well known that one can use Riemann theta functions to produce formulas for finite-gap solutions of NLS and the corresponding solutions of (67), which are known as *Baker eigenfunctions*, starting with a hyperelliptic Riemann surface Σ of genus g and certain other data (see [7], or Sections 4.1 and 4.3.2 in [5]). The solution (63) arises for $g = 1$, with the four branch points of Σ in conjugate pairs in the complex plane, so that the equation of Σ is

$$\mu^2 = (\lambda - \lambda_1)(\lambda - \overline{\lambda_1})(\lambda - \lambda_2)(\lambda - \overline{\lambda_2}), \tag{70}$$

where $\operatorname{Re}\lambda_1 < \operatorname{Re}\lambda_2$. (See Section 3 of [7] for a derivation of these solutions from general finite-gap formulas.) In genus one, the finite-gap solutions of (67), take the form

$$\psi_1 = \exp\left(\mathrm{i}\left(\Omega_1(P) - \frac{E}{2}\right)x + \mathrm{i}\left(\Omega_2(P) + \frac{N}{2}\right)t\right)\Theta_1(\xi, P),$$
$$\psi_2 = \exp\left(\mathrm{i}\left(\Omega_1(P) + \frac{E}{2}\right)x + \mathrm{i}\left(\Omega_2(P) - \frac{N}{2}\right)t\right)\Theta_2(\xi, P),$$

where E, N are constants determined by Σ, $P = (\lambda, \mu)$ is an arbitrary point on Σ that projects to $\lambda \in \mathbb{C}$, $\Omega_{1,2}$ are certain Abelian integrals on Σ, and $\Theta_{1,2}$ will be defined below. For the moment, we note that $\Theta_{1,2}$ depend only on P and $\xi = x - ct$ (where c is determined by the branch points of Σ) and are T-periodic in ξ.

After the substitutions (69) the squared Baker eigenfunctions yield solutions to (65) of the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\alpha t}\psi_1^2 \\ \mathrm{e}^{-\mathrm{i}\alpha t}\psi_2^2 \end{bmatrix} = \mathrm{e}^{2\mathrm{i}\xi\Omega_1(P)+2\mathrm{i}t(\Omega_2(P)+c\Omega_1(P))}\begin{bmatrix} \mathrm{e}^{\mathrm{i}(N-cE+\alpha)t-\mathrm{i}E\xi}\,\Theta_1(\xi, P)^2 \\ \mathrm{e}^{\mathrm{i}(cE-N-\alpha)t+\mathrm{i}E\xi}\,\Theta_2(\xi, P)^2 \end{bmatrix}.$$

In particular, these v_1, v_2 satisfy the ansatz (66) for $\alpha = cE - N$ and

$$\ell = 2\mathrm{i}(\Omega_2(P) + c\Omega_1(P)), \tag{71}$$

with

$$w_1(\xi) = \mathrm{e}^{\mathrm{i}(2\Omega_1(P)-E)\xi}\,\Theta_1(\xi; P)^2, \qquad w_2(\xi) = \mathrm{e}^{\mathrm{i}(2\Omega_1(P)+E)\xi}\,\Theta_2(\xi; P)^2. \tag{72}$$

Lemma 2. *The formula (71) implies that $\ell = 4\mathrm{i}\mu$.*

Proof. On the surface Σ, which has genus 1, we choose a basis $\{a, b\}$ for homology cycles, as shown in Fig. 6. The differentials $\mathrm{d}\Omega_1, \mathrm{d}\Omega_2$ have zero a-period. Denoting their b-periods as V, W respectively, from [7] we have

$$W/V = -c = (\lambda_1 + \lambda_2 + \overline{\lambda_1} + \overline{\lambda_2}).$$

(Note that the meaning of c here is -1 times its definition in [7].) Thus, the differential

$$\mathrm{d}\ell = 2\mathrm{i}(\mathrm{d}\Omega_2 + c\,\mathrm{d}\Omega_1)$$

has zero a- and b-periods. So, although $\Omega_{1,2}$ are not well defined on Σ, ℓ is a well-defined function on Σ.

The integrals Ω_i use the branch point $\overline{\lambda_2}$ as basepoint, so $\ell = 0$ there. Because $\iota^*\mathrm{d}\Omega_i = -\mathrm{d}\Omega_i$ for $i = 1, 2$ (where ι is the sheet exchange automorphism $(\lambda, \mu) \mapsto (\lambda, -\mu)$), then $\iota^*\ell = -\ell$. It follows that ℓ vanishes at all four branch points, which are the same points at which the coordinate μ vanishes. The surface Σ has two points at infinity, called $\infty_\pm$, where both λ and μ approach complex infinity and

$$\frac{\mu}{\lambda^{g+1}} = \frac{\mu}{\lambda^2} = \pm\left(1 + \frac{c}{2}\lambda^{-1} + \mathcal{O}(\lambda^{-2})\right)$$

respectively. Combining the asymptotic expansions for Ω_1 and Ω_2 gives

$$\frac{\ell}{\lambda^2} = \pm 2\mathrm{i}\left(2 + c\lambda^{-1} + \mathcal{O}(\lambda^{-2})\right).$$

Therefore, $\ell/\mu = 4\mathrm{i} + \mathcal{O}(\lambda^{-2})$. Because ℓ/μ is bounded and holomorphic on Σ, then it must be equal to the constant $4\mathrm{i}$, i.e., $\ell = 4\mathrm{i}\mu$. In other words, we have established the identity

$$\Omega_2(P) = -c\Omega_1(P) + 2\mu \tag{73}$$

for genus one NLS solutions. □

A.2. Forming a basis of solutions

Lemma 2 and (70) imply that for generic values of ℓ there are four distinct points $P \in \Sigma$ for which the Baker eigenfunctions at P may be used to construct a solution of the eigenvalue problem (6). The exceptional values are those for which two roots of (70), as a polynomial equation for λ, coincide.

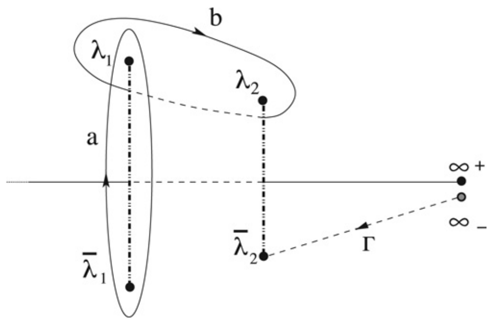

Fig. 6. Homology cycles and other integration paths on the genus one Riemann surface with complex conjugate branch points. Solid curves appear on the upper sheet, dashed curves on the lower sheet, and branch cuts extend between each branch point and its conjugate. The horizontal line represents a set of points where μ is real and positive.

From here on, we specialize to the case where the branch points in (70) satisfy $\lambda_2 = -\overline{\lambda_1}$, with $\operatorname{Re}\lambda_1 \neq 0$, which yields the NLS solution (63). In this case, $c = 0$ and $E = V/2$, the branch points are related to the elliptic modulus by $\lambda_1/|\lambda_1| = -k' + \mathrm{i}k$, and the finite-gap solution coincides with (63) for $\delta = 2|\lambda_1|$. The exceptional values of μ are

$$\mu = \pm|\lambda_1|^2, \qquad \mu = \pm\operatorname{Im}(\lambda_1^2), \tag{74}$$

corresponding respectively to a double root occurring at $\lambda = 0$, or two pairs of roots coinciding at opposite points along the real axis (if $k^2 < 1/2$) or imaginary axis (if $k^2 > 1/2$).

When μ is not an exceptional value, we will construct a matrix solution $W(x;\ell)$ of (8) where each column is of the form $[w_1(x), w_2(x), w_1'(x), w_2'(x)]^t$ and w_1, w_2 are given by (72) for one of the four points $P = (\lambda, \mu)$ satisfying (70) for the given μ. For purposes of showing that the matrix solution is nonsingular, it suffices to evaluate its determinant at one value of x.

We now need to specify the form of the factors Θ_1, Θ_2 in (72). Specializing the formulas given in Section 5 of [7], and using $\xi = x - ct = x$, we have

$$\Theta_1(x;P) = \frac{\theta(A(P) + \mathrm{i}Vx - D)\theta(D)}{\theta(\mathrm{i}Vx - D)\theta(A(P) - D)}, \qquad \Theta_2(x;P) = -\mathrm{i}e^{\Omega_3(P)}\frac{\theta(A(P) + \mathrm{i}Vx - D - R)\theta(D)}{\theta(\mathrm{i}Vx - D)\theta(A(P) - D)}.$$

In these formulas,

- $V = \pi|\lambda_1 - \overline{\lambda_2}|/K = 2\pi|\lambda_1|/K$;
- $A(P)$ is the value of the Abel map, obtained by integrating the differential $\omega = V\mathrm{d}\lambda/(2\mu)$ from ∞_- to P on Σ;
- $\Omega_3(P)$ is the integral, from $\overline{\lambda_2}$ to P, of the unique meromorphic differential $\mathrm{d}\Omega_3$ on P which has zero a-period and satisfies $\mathrm{d}\lambda_1 \sim \pm\lambda^{-1}\mathrm{d}\lambda$ near $\infty_\pm$;
- R is minus the period of $\mathrm{d}\Omega_3$ on the cycle b (see Fig. 6), and is also equal to $A(\infty_+)$ when the path of integration for the Abel map is chosen to avoid the homology cycles a and b (see Section 5 in [7]); in the case when $c = 0$, $R = \pi(K'/K + \mathrm{i})$ (see Section 3 in [7]);
- D is an arbitrary constant which is pure imaginary;
- $\theta(z)$ is the Riemann theta function with period $2\pi\mathrm{i}$ and quasiperiod $B = -2\pi K'/K$. It is related to the Jacobi theta functions of modulus k by $\theta(z) = \theta_3(z/(2\mathrm{i}))$.

In genus one, the constant D has no significance, as it can be absorbed through a shift in x. Thus, we may assume that $D = 0$ in the above formulas. Once this is done, the columns of W for $x = 0$ take the form

$$\begin{bmatrix} w_1(0) \\ w_2(0) \\ w_1'(0) \\ w_2'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -f(P)^2 \\ 2\mathrm{i}g(P) \\ -2\mathrm{i}f(P)^2 h(P) \end{bmatrix}, \tag{75}$$

where

$$f(P) \equiv e^{\Omega_3(P)}\frac{\theta(A(P) - R)}{\theta(A(P))}, \qquad g(P) \equiv \Omega_1(P) + V\Theta(A(P)) - \frac{E}{2},$$
$$h(P) \equiv \Omega_1(P) + V\Theta(A(P) - R) + \frac{E}{2},$$

and $\Theta(z) = \theta'(z)/\theta(z)$ is the logarithmic derivative of the Riemann theta function. (Note that $A(P)$ and $\Omega_i(P)$ are not individually well defined on Σ, because of the nonzero periods of the corresponding differentials along the homology cycles. We take the convention that the paths of integration for these differ by a fixed path Γ (shown in Fig. 6) from ∞_- to $\overline{\lambda_2}$ in Σ_0, where Σ_0 denotes the simply-connected domain that results from cutting Σ along the homology cycles. With this convention, f, g, h are well-defined meromorphic functions on Σ.)

The Riemann surface Σ has two holomorphic involutions, namely $\iota : (\lambda, \mu) \mapsto (\lambda, -\mu)$ and $\sigma : (\lambda, \mu) \mapsto (-\lambda, \mu)$. As $\iota^* d\Omega_i = -d\Omega_i$, $\iota^*\omega = -\omega$ and ι fixes the basepoint for Ω_i, $i = 1, 2, 3$, it follows that

$$f(\iota P) = 1/f(P), \qquad g(\iota P) = -h(P).$$

As $\sigma^* d\Omega_1 = -d\Omega_1$, $\sigma^* d\Omega_3 = d\Omega_3 - \omega$, $\sigma^*\omega = -\omega$, and $E = V/2$, it follows that

$$f(\sigma P) = f(P), \qquad g(\sigma P) = -g(P).$$

Let P_1, P_2, P_3, P_4 be four points on Σ corresponding to a given (nonexceptional) value of μ. Without loss of generality, we may assume that $P_2 = \sigma(P_1)$ and $P_4 = \sigma(P_3)$. Using the above formulas for the behavior of f, g, h under σ, we find that the matrix with columns given by (75) is

$$W(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -f(P_1)^2 & -f(P_1)^2 & -f(P_3)^2 & -f(P_3)^2 \\ 2ig(P_1) & -2ig(P_1) & 2ig(P_3) & -2ig(P_3) \\ -2if(P_1)^2h(P_1) & 2if(P_1)^2h(P_1) & -2if(P_3)^2h(P_3) & 2if(P_3)^2h(P_3) \end{bmatrix}. \tag{76}$$

Lemma 3. *The functions g, h satisfy the identity*

$$-g(P) = h(P) = \lambda, \qquad P = (\lambda, \mu) \in \Sigma.$$

Proof. The poles of $g(P) = \Omega_1(P) + \Theta(A(P)) - E/2$ can only occur where Ω_1 has a pole or $\theta(A(P)) = 0$. (We now work on the cut Riemann surface Σ_0.) These occur only at ∞_- (near which $\Omega_1 \sim -\lambda + \mathcal{O}(1)$, but $\theta(A(P)) = \theta(0) = \sqrt{2K/\pi} \neq 0$), and at ∞_+ (near which $\Omega_1 \sim \lambda + \mathcal{O}(1)$ and $\theta(A(P)) = \theta(R) = 0$). Near ∞_+, use $w = \lambda^{-1}$ as local coordinate. Then $A(P) = \frac{V}{2}\int_{\infty_-}^{P} \mu^{-1} d\lambda$ implies that

$$\frac{dA(P)}{dw} = -\frac{V}{2w^2\mu} = -\frac{V}{2}(1 + \mathcal{O}(w^2))$$

and therefore $A(P) = R - \frac{V}{2}w + \mathcal{O}(w^3)$. As $\theta(z)$ has a simple zero at $z = R$,

$$g(P) = \Omega_1(P) + V\frac{\theta'(A(P))}{\theta(A(P))} - \frac{E}{2} = w^{-1} + V\left(\frac{-2}{V}\right)w^{-1} + \mathcal{O}(1) = -\lambda + \mathcal{O}(1).$$

The oddness of g with respect to σ implies that $g(P) = 0$ when P lies over the origin in the λ-plane. Therefore, the quotient $g(P)/\lambda$ is a bounded holomorphic function on Σ, and so must be a constant. The above asymptotic expansion for g implies that $g(P) = -\lambda$, and it follows from the fact that $h(\iota P) = -g(P)$ that $h(P) = \lambda$. □

Taking Lemma 3 into account, we compute

$$\det W(0) = -16\lambda(P_1)\lambda(P_3)(f(P_1)^2 - f(P_3)^2)^2. \tag{77}$$

Lemma 4. *For μ not equal to zero or any of the exceptional values in (74), the matrix $W(0)$ is nonsingular. (We conjecture that the analogous statement is true for genus one finite-gap solutions in general.)*

Proof. As we are excluding the exceptional values, neither of the λ-values in (77) are zero, and it only remains to establish that $f(P_1) \neq \pm f(P_3)$.

Define $F(P) = f(P) + 1/f(P)$, which is a well-defined meromorphic function of λ. The formula for $f(P)$ shows that its only pole is a second-order pole at ∞_+ (because $\exp(\Omega_3(P)) \sim \lambda$ near there), and hence F is a second-order polynomial in λ. It is known that $f(\pm\lambda_1) = 1$ and $f(\pm\overline{\lambda_1}) = -1$ (see Section 7 of [8]). Hence,

$$F(\lambda) = \frac{4(\lambda^2 - d)}{\lambda_1^2 - \overline{\lambda_1}^2}, \quad d \equiv \mathrm{Re}\,(\lambda_1^2).$$

Using the quadratic formula, and the fact that f has a second-order zero at ∞_-, we obtain

$$f(P) = \frac{2(\lambda^2 - d + \mu)}{\lambda_1^2 - \overline{\lambda_1}^2}. \tag{78}$$

Let $y_1 = \lambda(P_1)$ and $y_3 = \lambda(P_3)$. As y_1 and y_3 are distinct roots of

$$(\lambda^2 - \lambda_1^2)(\lambda^2 - \overline{\lambda_1}^2) = \mu^2, \tag{79}$$

and $y_1 \neq -y_3$, then $y_1^2 + y_3^2 = 2d$. Hence, $F(y_1) = 2(y_1^2 - y_3^2) = -F(y_3)$, and therefore $f(P_1) \neq f(P_3)$. Furthermore, using (78), we see that

$$(\lambda_1^2 - \overline{\lambda_1}^2)(f(P_1) + f(P_3)) = 2(y_1^2 + y_3^2 - 2d + 2\mu) = 4\mu.$$

Thus, because we assume that $\mu \neq 0$, $f(P_1) \neq -f(P_3)$. □

Assuming that μ is not zero or one of the exceptional values, we can define the transfer matrix

$$N = W(T)W(0)^{-1},$$

whose eigenvalues describe the growth of solutions to (8) over one period. (In particular, there is a periodic solution if and only if N has eigenvalue one.) Using the fact that $T = 4\pi/V = 2\pi/E$ and $\Omega_1(\sigma P) = -\Omega_1(P) + E/2$ on the cut surface, we calculate that

$$W(T) = W(0)\begin{bmatrix} e^{2i\Omega_1(P_1)T} & 0 & 0 & 0 \\ 0 & e^{-2i\Omega_1(P_1)T} & 0 & 0 \\ 0 & 0 & e^{2i\Omega_1(P_3)T} & 0 \\ 0 & 0 & 0 & e^{-2i\Omega_1(P_3)T} \end{bmatrix}. \tag{80}$$

Then the characteristic polynomial of the transfer matrix is

$$\det(N - \tau I) = \Big((\tau+1)^2 - 4\tau\cos(\Omega_1(P_1)T)\Big)\Big((\tau+1)^2 - 4\tau\cos(\Omega_1(P_3)T)\Big).$$

By substituting $\tau = 1$ in this formula, we obtain

Lemma 5. *For μ not equal to zero or the exceptional values in (74), the system (8), where $\ell = 4i\mu$, has a nontrivial solution of period T if and only if* $\sin(\Omega_1(P_1)T) = 0$ *or* $\sin(\Omega_1(P_3)T) = 0$.

A.3. Imaginary zeros of the Evans function

For periodic finite-gap solutions of NLS, the *Floquet discriminant* [7] takes the form

$$\Delta(\lambda) = 2\cos\left(\frac{2\pi}{V}\Omega_1(P)\right) = 2\cos\left(\frac{T}{2}\Omega_1(P)\right).$$

(While Ω_1 is odd with respect to the involution ι, the evenness of the cosine makes Δ a well-defined function of λ.) This has the property that the AKNS system (67) admits a T-periodic (respectively, antiperiodic) solution if and only $\Delta = \pm 2$.

The consequence of Lemmas 4 and 5 is that, for values of ℓ that correspond to nonzero values of μ excluding those in (74), the Evans function is equal to zero if and only if $\Delta = \pm 2$ or $\Delta = 0$ at an opposite pair out of the four λ-values corresponding to μ. When this happens, the corresponding pair of columns of W are periodic, and the corresponding value of ℓ is a zero of the Evans function of geometric multiplicity two. Thus, we can use the Floquet discriminant to find the (nonexceptional) zeros of the Evans function.[3]

In Section 3.5 of [7], the discriminant for genus one finite-gap solutions is calculated as

$$\Delta(\lambda) = -2\cos\left[2iK\left(Z(u) - \beta^2\frac{\operatorname{cn} u \operatorname{dn} u \operatorname{sn} u}{1 - \beta^2 \operatorname{sn}^2 u}\right)\right], \tag{81}$$

where the variable u is related to λ by

$$\operatorname{sn}^2 u = \varphi(\lambda) \equiv \frac{(\lambda_1 - \overline{\lambda}_2)(\lambda - \overline{\lambda}_1)}{(\lambda_1 - \overline{\lambda}_1)(\lambda - \overline{\lambda}_2)} \tag{82}$$

and the parameter β and the modulus k are related to the branch points by

$$k^2 = \frac{-(\lambda_1 - \overline{\lambda}_1)(\lambda_2 - \overline{\lambda}_2)}{|\lambda_1 - \overline{\lambda}_2|^2}, \qquad \beta^2 = \frac{\lambda_1 - \overline{\lambda}_1}{\lambda_1 - \overline{\lambda}_2}.$$

[3] Even if one of the double points gives an exceptional value of μ, it still gives a zero of the Evans function. However, there may be additional zeros at exceptional values of μ, corresponding to possible periodic solutions of (8) obtained from solutions of the form (72) by reduction of order.

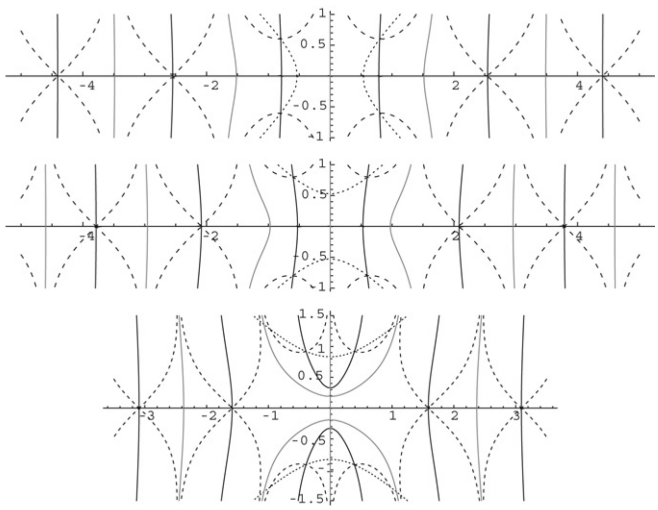

Fig. 7. Level curves of the Floquet discriminant Δ in the complex λ-plane for elliptic moduli $k = .6$, $k = .8$ and $k = .92$, respectively. (We take $|\lambda_1| = \delta/2 = 1$.) In each of the diagrams, the solid curves indicate where Δ is real, the gray curves where Δ is imaginary, the dashed curves where $\mathrm{Re}\,\Delta = \pm 2$, and the dotted curves where μ is real (which includes the real and imaginary axes). Thus, branch points occur where dotted intersects dashed, periodic points occur where solid intersects dashed, and points where $\Delta = 0$ occur where solid intersects gray. The continuous Floquet spectrum consists of the real axis and two bands, which are the parts of the two central solid curves bounded by their intersections with dashed curves.

In the symmetric case (i.e., $\lambda_2 = -\overline{\lambda}_1$) which we are considering here, these specialize to

$$-k' + \mathrm{i}k = \lambda_1/|\lambda_1|, \qquad \beta^2 = k(k - \mathrm{i}k'),$$

and the variable u is determined by λ as follows. The function $\mathrm{sn}^2 u$ has periods $2K$ and $2\mathrm{i}K'$, and is even in u, so it suffices to restrict u to the set $(0, K] \times (-K', K']$ in the complex plane. Then there is a unique u in this set for each λ in the extended complex plane. Conversely, solving (82) for λ in terms of u gives

$$\lambda(u) = \frac{\lambda_1 \mathrm{sn}^2 u + \overline{\lambda}_1 \beta^{-2}}{\beta^{-2} - \mathrm{sn}^2 u}.$$

The locus of λ-values for which $\Delta(\lambda)$ is real and between 2 and -2 is known as the *continuous Floquet spectrum* of the NLS potential q. For genus one solutions, it consists of the real axis and two bands terminating at the branch points. (For low values of k the bands emerge from the real axis, but for k sufficiently near 1, the bands become detached from the real axis (see the solid curves in Fig. 7); the transition occurs around $k \simeq .9089$ [20].)

The periodic points (i.e., where $\Delta = \pm 2$) consist of the branch points and countably many points (known as double points) on the real axis. Points where $\Delta = 0$ will naturally interlace the periodic points along the real axis, but will also occur along the imaginary axis if the bands are detached. When this happens, though, the corresponding value of μ is real. Hence, all the zeros of the Evans function will occur along the imaginary axis in the complex ℓ-plane.

Figs. 1 and 7 show the zeros of Evans function in the complex ℓ-plane and the level sets of Δ in the λ-plane for several different moduli. Fig. 1 also includes the locus of ℓ-values which are related by (79), with $\ell = 4\mathrm{i}\mu$, to λ-values in the continuous spectrum. In fact, we now prove Theorem 2, which states that the system (8) has a nontrivial solution that is bounded for all $x \in \mathbb{R}$ if and only if ℓ is related by (79) to a point of the continuous Floquet spectrum.

Proof of Theorem 2. Clearly, if ℓ is related to a point λ belonging to the continuous Floquet spectrum, then one of the solutions constructed in Appendix A.2 will be bounded.

For the converse, suppose that $\boldsymbol{\omega}(x)$ is a nontrivial solution of (8) for a particular ℓ, and $\boldsymbol{\omega}(x)$ is bounded for all x. If ℓ is one of the exceptional values (or zero) then the λ-values given by (79) already belong to the continuous Floquet spectrum, so we may assume that $\ell \neq 0$ and is not an exceptional value. Thus, our solution will be of the form $\boldsymbol{\omega}(x) = W(x)\boldsymbol{c}$, where $W(x)$ is the fundamental solution matrix constructed in Appendix A.2 and $\boldsymbol{c} = (c_1, c_2, c_3, c_4)^{\mathrm{T}}$ is a nonzero vector of constants. Evaluating after n periods and using (80) gives

$$\boldsymbol{\omega}(nT) = W(0) \begin{bmatrix} c_1 \mathrm{e}^{2n\mathrm{i}\Omega_1(P_1)T} \\ c_2 \mathrm{e}^{-2n\mathrm{i}\Omega_1(P_1)T} \\ c_3 \mathrm{e}^{2n\mathrm{i}\Omega_1(P_3)T} \\ c_4 \mathrm{e}^{-2n\mathrm{i}\Omega_1(P_3)T} \end{bmatrix}.$$

Using (76) shows that the top entry of this vector is

$$c_1 \mathrm{e}^{2n\mathrm{i}\Omega_1(P_1)T} + c_2 \mathrm{e}^{-2n\mathrm{i}\Omega_1(P_1)T} + c_3 \mathrm{e}^{2n\mathrm{i}\Omega_1(P_3)T} + c_4 \mathrm{e}^{-2n\mathrm{i}\Omega_1(P_3)T}$$

while the next entry is

$$-f(P_1)^2 \left(c_1 \mathrm{e}^{2n\mathrm{i}\Omega_1(P_1)T} + c_2 \mathrm{e}^{-2n\mathrm{i}\Omega_1(P_1)T}\right) - f(P_3)^2 \left(c_3 \mathrm{e}^{2n\mathrm{i}\Omega_1(P_3)T} + c_4 \mathrm{e}^{-2n\mathrm{i}\Omega_1(P_3)T}\right). \quad (83)$$

As $f(P_1)^2 \neq f(P_3)^2$ (by the proof of Lemma 4 above) these entries cannot both be bounded for all n unless each factor in (83) is bounded. Considering the factor $c_1 \mathrm{e}^{2n\mathrm{i}\Omega_1(P_1)T} + c_2 \mathrm{e}^{-2n\mathrm{i}\Omega_1(P_1)T}$ shows that it cannot be bounded for all n unless either c_1 and c_2 are both zero, or $\mathrm{Im}\,\Omega_1(P_1) = 0$. Similar considerations for the other factor, and the fact that $\boldsymbol{c} \neq 0$, yield that either $\mathrm{Im}\,\Omega_1(P_1) = 0$ or $\mathrm{Im}\,\Omega_1(P_3) = 0$. Thus, one of the λ-values given by (79) must belong to the continuous Floquet spectrum. □

Appendix B. Analyticity of the emerging eigenvalues

In this appendix, we prove the following theorem.

Theorem 5. *The linear operator $\mathcal{L}$ defined in (42) has two distinct nonzero eigenvalues counting multiplicity which, for small values of ϵ, are $\mathcal{O}(\epsilon)$ and analytic functions of ϵ*

Lemma 6. *Let $F(x, y)$ be a complex-valued function of the two complex variables x and y which is analytic around $(0, 0) \in \mathbb{C}^2$. Suppose furthermore that $F_x(0, 0) = F_y(0, 0) = F(0, 0) = 0$, $F_{yy}(0, 0) \neq 0$, and that the Hessian determinant of F is not zero at $(0, 0)$. Then for small values of x the equation $F(x, y) = 0$ has two distinct solutions $y = f_k(x)$, $k = 1, 2$ such that $f_k(0) = 0$ but $f_1'(0) \neq f_2'(0)$. Furthermore, these two solutions are analytic in a neighborhood of $x = 0$.*

Proof. Using the Weierstrass preparation theorem [32], for small x and y, F can be written as

$$F(x, y) = \left(y^2 + g_1(x)y + g_0(x)\right) h(x, y),$$

where g_i and h are analytic, $h(0, 0) \neq 0$, and $g_i(0) = 0$. Furthermore, since $F_x(0, 0) = 0$, we have that $g_0'(0) = 0$. Thus, the two solutions of the equation $F(x, y) = 0$ around $(0, 0)$ take the form

$$y = \frac{-g_1(x) \pm \sqrt{\widetilde{g}(x)}}{2}, \quad \widetilde{g}(x) = g_1(x)^2 - 4g_0(x). \quad (84)$$

Using the definition of $\widetilde{g}$ above, we have that $\widetilde{g}(0) = \widetilde{g}'(0) = 0$ and $\widetilde{g}''(0) = 2\left(g_1'(0)^2 - 2g_0''(0)\right)$. Furthermore, the Hessian determinant of F at $(0, 0)$ is given by

$$F_{xy}(0, 0)^2 - F_{xx}(0, 0)F_{yy}(0, 0) = \left(g_1'(0)^2 - 2g_0''(0)\right) h(0, 0) = \frac{h(0, 0)^2}{2}\widetilde{g}''(0).$$

Thus, using our assumption about the Hessian, we have that $\widetilde{g}''(0) \neq 0$ and the function $\widetilde{g}$ can be written, for small values of x, as $\widetilde{g}(x) = x^2\widetilde{m}(x)$, for some analytic function $\widetilde{m}$ such that $\widetilde{m}(0) \neq 0$. After inserting this expression for $\widetilde{g}$ into (84), it is easy to see that the two solutions in (84) are analytic around $x = 0$. □

In the proof above, let us consider the case where $F_{xy}(0, 0)^2 - F_{xx}(0, 0)F_{yy}(0, 0) = 0$. If $\widetilde{g}(x)$ is identically zero, then by (84) there is only one analytic solution. If $\widetilde{g}$ is not identically zero, we can state that, for small values of x, $\widetilde{g}(x) = x^k\widetilde{m}(x)$, for some analytic function $\widetilde{m}$ such that $\widetilde{m}(0) \neq 0$ and some integer $k > 2$. If k is even, then again we can say that there are two well-defined analytic functions $y = f_1(x)$ and $y = f_2(x)$ which coincide up to and including first order in x at $x = 0$. If k is odd, fix a branch of $\sqrt{\widetilde{m}(x)}$ near $x = 0$. Then

$$y = f_1(x) = \frac{-g_1(x) + x^{k/2}\sqrt{\widetilde{m}(x)}}{2}, \qquad y = f_2(x) = \frac{-g_1(x) - x^{k/2}\sqrt{\widetilde{m}(x)}}{2}.$$

In order to make f_1 and f_2 well defined, we have to take a branch cut (along the negative real x-axis let say). Then in a disc, minus this slit, f_1 and f_2 have series expansions involving half-integer powers in x which are absolutely convergent for small $|x|$. We thus have the following lemma.

Lemma 7. *Let $F(x, y)$ satisfy the same conditions as in Lemma 6 except that we assume the Hessian determinant of F to be zero at $(0, 0)$. Then for small values of x the equation $F(x, y) = 0$ has one of the following:*

(a) *only one solution, $y = f(x)$ which is analytic in x;*

(b) *two distinct analytic solutions* $y = f_1(x)$ *and* $y = f_2(x)$ *whose series expansions at* $x = 0$ *are the same up to and including first order in* x*;*

(c) *two distinct solutions which, in a disc around the origin from which a slit is removed, can be expanded as* $y = f_k = \sum_{n=1}^{\infty} a_n^{(k)} x^{n/2}$, $k = 1, 2$, *with* $a_1^{(1)} = a_1^{(2)} = 0$ *and* $a_2^{(1)} = a_2^{(2)}$.

Let us now prove Theorem 5.

Proof. We rewrite the eigenvalue problem (42) arising from the linearized CGL as a first-order linear system, with T-periodic coefficients which involve ℓ:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} w_1 \\ w_2 \\ (1-\mathrm{i}\epsilon)w_1' \\ (1+\mathrm{i}\epsilon)w_2' \end{bmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -[\alpha + \mathrm{i}(\ell - \epsilon r) + 4|U|^2(1+\mathrm{i}\epsilon s)] & -2(1+\mathrm{i}\epsilon s)U^2 & 0 & 0 \\ -2(1-\mathrm{i}\epsilon s)\overline{U}^2 & -[\alpha - \mathrm{i}(\ell - \epsilon r) + 4|U|^2(1-\mathrm{i}\epsilon s)] & 0 & 0 \end{pmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_1' \\ w_2' \end{bmatrix}. \quad (85)$$

In the same way as the Evans function M_0 was defined in (11) for the system (8), we define the Evans function for (85) to be a function of ℓ and ϵ that we denote by $M_1(\ell, \epsilon)$.

Note that M_1 is an analytic function of ℓ and ϵ. For a given ϵ, the zeros of M_1 in the ℓ complex plane are the elements of the point spectrum of $\mathcal{L}$. Furthermore, $M_1(\ell, 0) = M_0(\ell)$ because (8) corresponds to the system (85) for $\epsilon = 0$. For $\epsilon \neq 0$, the zero eigenvalue has multiplicity two and thus $M_1 = \mathcal{O}(\ell^2)$ as ℓ approaches zero and $N \equiv M_1/\ell^2$ is a well-defined function for small ℓ and ϵ. The emerging eigenvalues are then found by considering the solution set of the implicit equation

$$N(\ell, \epsilon) = 0.$$

In order to apply the Lemma 6 to the function N with $\epsilon = x$ and $\ell = y$, we need to verify that $N_\epsilon(0, 0) = M_{1\ell\ell\epsilon}(0, 0)/2 = 0$. This can be proven as follows, using a similar argument as the one used for the proof of Theorem 3. Let $\widetilde{\mathbf{w}}_i(x; \ell, \epsilon)$, $i = 1 \ldots 4$, be four linearly independent solutions of (85) with initial conditions $\widetilde{\mathbf{w}}_i(0; \ell, \epsilon) = A_i$ where A_i denotes the ith column of A defined in (26). These solutions can be expanded in ϵ in the following way

$$\widetilde{\mathbf{w}}_i(x; \ell, \epsilon) = \mathbf{w}_i(x; \ell) + \epsilon \widetilde{\mathbf{w}}_i^1(x; \ell) + \epsilon^2 \widetilde{\mathbf{w}}_i^2(x; \ell) + \mathcal{O}(\epsilon^3), \quad (86)$$

where the $\mathbf{w}_i(x; \ell)$ are the solutions of (8) defined in (19). Let $\widetilde{W}(x; \ell)$ be a fundamental matrix solution of (85) with columns $\widetilde{\mathbf{w}}_i$. We then define

$$\widetilde{M}_1(\ell, \epsilon) = \det\left(\widetilde{W}(T; \ell, \epsilon) - A\right),$$

where A is defined in (26). Then,

$$M_1(\ell, \epsilon) = \frac{1}{4\delta^6 k^4} \widetilde{M}_1(\ell, \epsilon). \quad (87)$$

In what follows, we prove that $\widetilde{M}_{1\ell\ell\epsilon}(0, 0) = 0$. Using the same argument as the one used to obtain (27), we get

$$\frac{1}{2}\widetilde{M}_{1\ell\ell}(0, \epsilon) = \det\left(\mathbf{w}_1^1(T) + \mathcal{O}(\epsilon), \mathbf{w}_2^1(T) + \mathcal{O}(\epsilon), \mathbf{V}_3^h(T) - A_3 + \mathcal{O}(\epsilon), \mathbf{V}_4^h(T) - A_4 + \mathcal{O}(\epsilon)\right). \quad (88)$$

The derivative $\widetilde{M}_{1\ell\ell\epsilon}(0, 0)$ is obtained by differentiating (88) with respect to ϵ and evaluating it at $\epsilon = 0$. By doing so, we obtain a sum of determinants of matrices each involving three of the four columns of the matrix in (27). Thus, we must have $\widetilde{M}_{1\ell\ell\epsilon}(0, 0) = 0$.

Assuming

$$N_{\ell\epsilon}(0, 0)^2 - N_{\ell\ell}(0, 0) N_{\epsilon\epsilon}(0, 0) \neq 0, \quad (89)$$

we can now apply the Lemma 6 to the function N and conclude that the emerging eigenvalues are analytic functions of ϵ and $\mathcal{O}(\epsilon)$ for small ϵ. In the case in which the condition (89) is not met, we only need to consider the situation in which the Hessian determinant is zero for all the possible values of the parameters k and s (the parameter δ is set to 1 without loss of generality). Indeed, since the left-hand side of (89) is an analytic function of k and s, it is either zero for all values of k and s or only on a subset of measure zero. In the latter case, the eigenvalues are analytic in k and s and are analytic in ϵ for almost all values k and s, so they must be analytic in ϵ for all values of k and s.

We now assume that the left-hand side of (89) is identically zero, with the aim of reaching a contradiction. Lemma 7 enables us to expand the emerging eigenvalues as

$$\ell = \sum_{n=2}^{\infty} \widetilde{\ell}_n \epsilon^{n/2} = \sum_{n=2}^{\infty} \widetilde{\ell}_n \zeta^n, \quad \text{where } \epsilon = \zeta^2, \quad (90)$$

for small ζ. The variable ζ was introduced to avoid the necessity of considering a disc with a slit around the origin. Substituting this series into (42) and expanding $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{\omega}}_n \zeta^n,$$

one finds, at orders 0, 1, 2, 3 and 4 in ζ, the following equations

$$\begin{aligned}
\mathcal{L}_0\widetilde{\boldsymbol{\omega}}_0 &= 0,\\
\mathcal{L}_0\widetilde{\boldsymbol{\omega}}_1 &= 0,\\
\mathcal{L}_0\widetilde{\boldsymbol{\omega}}_2 &= \widetilde{\ell}_2\widetilde{\boldsymbol{\omega}}_0 - \mathcal{L}_1\widetilde{\boldsymbol{\omega}}_0,\\
\mathcal{L}_0\widetilde{\boldsymbol{\omega}}_3 &= \widetilde{\ell}_2\widetilde{\boldsymbol{\omega}}_1 + \widetilde{\ell}_3\widetilde{\boldsymbol{\omega}}_0 - \mathcal{L}_1\widetilde{\boldsymbol{\omega}}_1,\\
\mathcal{L}_0\widetilde{\boldsymbol{\omega}}_4 &= \widetilde{\ell}_3\widetilde{\boldsymbol{\omega}}_1 + \widetilde{\ell}_4\widetilde{\boldsymbol{\omega}}_0 + \widetilde{\ell}_2\widetilde{\boldsymbol{\omega}}_2 - \mathcal{L}_1\widetilde{\boldsymbol{\omega}}_2 - \mathcal{L}_2\widetilde{\boldsymbol{\omega}}_0,
\end{aligned} \tag{91}$$

where $\mathcal{L}_i$ are the coefficients of the expansion $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \mathcal{O}(\epsilon^3)$. To find a solvability condition for (91), we apply Lemma 1 to the first, third, and fifth equation with $\widetilde{\boldsymbol{\omega}}_0$, $\widetilde{\boldsymbol{\omega}}_2$, and $\widetilde{\boldsymbol{\omega}}_4$ replacing $\boldsymbol{\omega}_0$, $\boldsymbol{\omega}_1$, and $\boldsymbol{\omega}_2$, respectively. Thus, like ℓ_1, $\widetilde{\ell}_2$ is zero or is given by either one of the expressions in (51). However, this latter condition does not generically give rise to two equal values of ℓ_1 in (51) (see Fig. 3 where the zero sets of the two quantities in (51) are plotted). Furthermore, it is easy to show that once $\widetilde{\ell}_2$ in the series (90) is known, then the subsequent $\widetilde{\ell}_n$ are uniquely determined by solvability conditions at higher orders. In view of the last two facts, cases (b) and (c) of Lemma 7, where the two roots are distinct but the same at order ϵ, cannot hold.

We are thus led to consider case (a), where there is only one emerging eigenvalue analytic in ϵ for all values of k and s. As in Section 3.2, we begin by expanding the eigenvalue ℓ and the corresponding eigenvector $\boldsymbol{\omega}$ as

$$\ell = \sum_{n=1}^{\infty} \ell_n \epsilon^n, \qquad \boldsymbol{\omega} = \sum_{n=0}^{\infty} \boldsymbol{\omega}_n \epsilon^n.$$

From the computations performed in Section 3.2, we have that $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ are given by (52) and (53), i.e.,

$$\boldsymbol{\omega}_0 = \alpha\boldsymbol{\omega}_0^1 + \beta\boldsymbol{\omega}_0^2, \qquad \boldsymbol{\omega}_1 = \alpha\boldsymbol{\omega}_1^1 + \beta\boldsymbol{\omega}_1^2 + \ell_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right), \tag{92}$$

for some constants α and β. Furthermore, ℓ_1 must satisfy either one of the expressions of (51) which are derived from a solvability condition. In addition to that, since we are considering the case in which the two solutions coincide and since the order of the zero of the Evans function equals the algebraic multiplicity of the eigenvalue, there must be a generalized eigenvector $\boldsymbol{\omega}^g$ satisfying

$$\mathcal{L}\boldsymbol{\omega}^g = \ell\boldsymbol{\omega}^g + \boldsymbol{\omega}. \tag{93}$$

Expanding $\boldsymbol{\omega}^g$ analytically as

$$\boldsymbol{\omega}^g = \sum_{n=0}^{\infty} \boldsymbol{\omega}_n^g \epsilon^n,$$

and substituting in (93), one obtains, at orders zero and one, the following equations:

$$\mathcal{L}_0\boldsymbol{\omega}_0^g = \boldsymbol{\omega}_0, \tag{94}$$

$$\mathcal{L}_0\boldsymbol{\omega}_1^g = \ell_1\boldsymbol{\omega}_0^g + \boldsymbol{\omega}_1 - \mathcal{L}_1\boldsymbol{\omega}_0^g. \tag{95}$$

The solution of (94) is

$$\boldsymbol{\omega}_0^g = \alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2} \tag{96}$$

for the same α and β. Then, using (92) and (96), (95) becomes

$$\mathcal{L}_0\boldsymbol{\omega}_1^g = 2\ell_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right) + \alpha\boldsymbol{\omega}_1^1 + \beta\boldsymbol{\omega}_1^2 - \mathcal{L}_1\left(\alpha\boldsymbol{\omega}_0^{g,1} + \beta\boldsymbol{\omega}_0^{g,2}\right). \tag{97}$$

The solvability condition for (97), which is obtained by taking the inner product of the right side with elements of the kernel of $\mathcal{L}_0^\dagger$ (the same way the solvability condition for (55) was found), implies that ℓ_1 is given by either one of the expressions

$$\ell_1 = \frac{\left\langle \mathcal{L}_1\boldsymbol{\omega}_0^{g,1}, J\boldsymbol{\omega}_0^1\right\rangle - \left\langle \boldsymbol{\omega}_1^1, J\boldsymbol{\omega}_0^1\right\rangle}{2\left\langle \boldsymbol{\omega}_0^{g,1}, J\boldsymbol{\omega}_0^1\right\rangle} \quad \text{or} \quad \ell_1 = \frac{\left\langle \mathcal{L}_1\boldsymbol{\omega}_0^{g,2}, J\boldsymbol{\omega}_0^2\right\rangle - \left\langle \boldsymbol{\omega}_2^1, J\boldsymbol{\omega}_0^2\right\rangle}{2\left\langle \boldsymbol{\omega}_0^{g,2}, J\boldsymbol{\omega}_0^2\right\rangle}. \tag{98}$$

These are each one half of the expressions in (51), one of which ℓ_1 must also be equal to. By examining Fig. 3 which shows the zero loci in the k–s plane for the expressions given in (51), it becomes clear that neither of the expressions of (51) is equal to half the value of the other for all k and s. Thus, ℓ_1 cannot simultaneously satisfy one equation in (51) and one in (98) for all values of k and s.

We thus obtain a contradiction, and conclude that the left-hand side of (89) cannot be identically zero. Hence, the emerging eigenvalues are distinct and analytic in ϵ. □

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