AFFINE ISOMETRIC EMBEDDINGS AND RIGIDITY

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Abstract The Pick cubic form is a fundamental invariant in the (equi)affine differential geometry of hypersurfaces. We study its role in the affine isometric embedding problem, using exterior differential systems (EDS). We give pointwise conditions on the Pick form under which an isometric embedding of a Riemannian manifold M^3 into $\mathbb{R}^4$ is rigid. The role of the Pick form in the characteristic variety of the EDS leads us to write down examples of non-rigid isometric embeddings for a class of warped product M^3's.

1. Introduction

A strictly convex hypersurface M in $\mathbb{R}^{n+1}$ can be given a Riemannian metric in a way that is invariant under affine motions, i.e., the action of $SL(n+1)$ and translations. Essentially, if f is the position in $\mathbb{R}^{n+1}$ as a function of local coordinates on M, let

$$h_{ij} = \det\left(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}, \frac{\partial^2 f}{\partial x^i \partial x^j}\right).$$

Because of convexity, the coordinates can be chosen so that h_{ij} is positive definite; then the metric is

$$g_{ij} = (\det h)^{\left(\dfrac{-1}{n+2}\right)} h_{ij}$$

(see [Ca]). Alternatively, given a transverse vector field N along the hypersurface, define a connection ∇, for vector fields X and Y along M, by splitting the ordinary derivative

$$D_X Y = g(X,Y)N + \nabla_X Y$$

into transverse and tangential parts. By wedge product with N, the invariant volume form on the ambient space gives a volume form Ω on M. Then N

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is the unique affine normal, and g is the affine metric, if Ω is parallel with respect to ∇ and coincides with the volume form of g (see [No]). The second formulation will allow us to define the metric using moving frames. Note that ∇ is not necessarily the Levi-Civita connection for g: the difference between the two is essentially the Pick form. The Pick form is a totally symmetric cubic form on the tangent space, which is traceless with respect to g. Quadric hypersurfaces are characterized by the vanishing of the Pick form.

In this article we will discuss the affine isometric embedding problem: that is, given a Riemannian metric g on M^n, what are the strictly convex embeddings of M into $\mathbb{R}^{n+1}$ such that the affine invariant metric coincides with g? From a naive point of view, asking for such an embedding amounts to asking $n+1$ functions—the components of f—to satisfy a system of $n(n+1)/2$ second-order partial differential equations. This is overdetermined when $n > 2$. Even when $n = 2$, the determined case, it is not known for which metrics on $M = S^2$ there exist solutions defined on all of M. Although the affine invariant metrics for quadric hypersurfaces in all dimensions have constant curvature, it is not known for $n > 2$ if this is even the only local way of embedding these metrics.

The question we will address in this article is: if a metric on M can be isometrically embedded, is it rigid? We will concentrate on the first overdetermined case, when $n = 3$. Our results depend on the $SO(3)$-invariant properties of the Pick form. For instance, once one complexifies and projectivizes the tangent space at a point of M, the cubic form defines a cubic curve in $\mathbb{CP}^2$ which we will call the Pick curve. Generically, one expects there to be six points on this curve that are isotropic (null) directions for the quadratic form obtained by complexifying the metric. However, for the EDS whose integral manifolds correspond to isometric embeddings, the characteristic variety is consists of the double points of this intersection. This fact leads to

Theorem 1. *For any Riemannian three-manifold, the set of affine isometric embeddings with whose Pick curve contains no double isotropic points is finite-dimensional.*

In standard coordinates on a three-dimensional inner product space, a traceless cubic form becomes a harmonic cubic polynomial. We will say that a harmonic cubic σ on $\mathbb{R}^3$ is $SO(3)$-generic if the linear map Φ from the space of harmonic quintics α on $\mathbb{R}^3$ to the space of 2-form-valued quadratics, given by

$$\Phi(\alpha) := \sum_{i,j} \frac{\partial^2 \, d\sigma}{\partial x^i \partial x^j} \wedge \frac{\partial^2 \, d\alpha}{\partial x^i \partial x^j}, \tag{1}$$

is injective. (As explained in §4, the map Φ is in fact the direct sum of two homomorphisms belonging to the Clebsch-Gordan decomposition for the tensor product of the $SO(3)$ modules containing σ and α.) Calculating the prolongations of the EDS leads to

Theorem 2. *An affine isometric embedding of a connected M^3 with $SO(3)$-generic Pick form is completely determined, up to affine motions, by the value of the Pick form and its covariant derivative at one point of M.*

This result should be compared with the affine analogue of Bonnet's theorem, which says that a traceless cubic form on a Riemannian manifold M^n, satisfying certain integrability conditions, is the Pick form for an affine isometric embedding of M into $\mathbb{R}^{n+1}$ which is unique up to affine motions (see [Si], e.g.). In other words, the Pick form as a tensor *field* determines the embedding, up to affine motions.

We should reassure the reader that if a harmonic cubic satisfies the hypotheses of Theorem 2, then it satisfies those of Theorem 1. For, if the curve defined by a harmonic cubic has double isotropic points, we can arrange by rotation that they are $[1, \pm i, 0]$, whereupon the cubic must have the form

$$\sigma = az(2z^2 - 3x^2 - 3y^2) + (bx + cy)(12z^2 - 3x^2 - 3y^2). \qquad (2)$$

One can also arrange, by rotation, that $c = 0$. (In fact, this shows that the space of cubics with double isotropic points is of codimension two in the space of all harmonic cubics in three variables.) Using this form for the cubic, it is easy to check that the map Φ has a two-dimensional kernel. However, not every cubic satisfying the hypotheses of Theorem 1 satisfies those of Theorem 2: neither of the harmonic cubics xyz and $\lambda x(4z^2 - x^2 - y^2) + \lambda z(4y^2 - x^2 - z^2)$ has double isotropic points, but the former is $SO(3)$-generic while the latter is not.

To have any hope that the space of integral manifolds of an EDS is more than finite-dimensional, the characteristic variety must be non-empty. This leads us to classify, using another EDS, the hypersurfaces for which the Pick form has the form (2) at each point. In particular, among the hypersurfaces for which the Pick curve is reducible, there are a set of hypersurfaces whose metrics are warped products of constant-curvature surfaces. This leads to:

Theorem 3. *A warped product $dt^2 + f(t)^2 ds^2$, where ds^2 is a constant curvature metric on a surface, has a 1-parameter family of local affine isometric embeddings which are distinct under affine motions.*

We will briefly outline the rest of the paper. In §2 we introduce moving frames and set up an EDS for affine isometric embedding. In §3 we compute the characteristic variety for the first prolongation of this system when $n = 3$, and prove Theorem 1. In §4 we show that the tableau for the third prolongation is empty in the generic case, and prove Theorem 2. In §5 we classify hypersurfaces for which the Pick form has the form (2), prove Theorem 3, and construct global isometric embeddings, as hypersurfaces of this type, for certain warped product metrics on S^3. We also include an appendix with two lemmas, used in §5, concerning linear tableaux and the "saturation" of linear Pfaffian systems.

2. The isometric embedding system

Let $\mathcal{F}$ be the affine frame bundle of $\mathbb{R}^{n+1}$, $n \geq 2$; an affine frame at a point in $\mathbb{R}^{n+1}$ is a basis $e_0, \cdots, e_n$ of the tangent space there, such that the vectors form the columns of a matrix with determinant one. On $\mathcal{F}$ we regard the basepoint projection x and $e_0, \cdots, e_n$ as $\mathbb{R}^{n+1}$-valued functions, and we have the 1-forms ω^α and ω^α_β defined by

$$dx = e_\alpha \omega^\alpha \qquad de_\alpha = e_\beta \omega^\beta_\alpha$$

and satisfying the structure equations

$$d\omega^\alpha = -\omega^\alpha_\beta \wedge \omega^\beta \qquad d\omega^\alpha_\beta = -\omega^\alpha_\gamma \wedge \omega^\gamma_\beta. \tag{3}$$

(We will use the summation convention, and the index ranges $0 \leq \alpha, \beta, \gamma \leq n$ and $1 \leq i, j, k \leq n$.) On $\mathcal{F}$ the only linear relation these forms satisfy is

$$\omega^0_0 + \cdots + \omega^n_n = 0. \tag{4}$$

A smooth convex hypersurface $M^n \subset \mathbb{R}^{n+1}$ can be equipped with a smooth framing such that $e_1, \cdots, e_n$ are tangent to the hypersurface and orthonormal for the affine invariant metric, and e_0 is the affine normal. (Of course, such a framing is not unique.) By definition, this gives a lift of M into $\mathcal{F}$ to which the 1-forms ω^0, ω^0_0 and $\omega^0_i - \omega^i$, for $1 \leq i \leq n$, restrict to be zero. By differentiating the last set of 1-forms, using (3), we see that

$$\omega^i_j + \omega^j_i = 2 s_{ijk} \omega^k, \qquad s_{ikj} = s_{ijk} \tag{5}$$

when restricted. The Pick form is $s_{ijk}\omega^i\omega^j\omega^k$; because of (4), it satisfies the tracelessness, or 'apolarity', condition $s_{ijj} = 0$. Differentiating $\omega^0_0 = 0$ shows that

$$\omega^i_0 = 2L_{ij}\omega^j, \qquad L_{ji} = L_{ij}. \tag{6}$$

The affine second fundamental form is $-2L_{ij}\omega^i\omega^j$; our normalizations here are chosen to avoid fractions later on. Let

$$\omega^i_j = \sigma^i_j + \tau^i_j \tag{7}$$

where $\sigma^i_j = \sigma^j_i$ and $\tau^i_j = -\tau^j_i$. Then (5) and the structure equations (3) show that τ^i_j are the Levi-Civita connection forms for the metric.

Now let M^n be a given Riemannian manifold, and let B be the orthonormal frame bundle of M, equipped with canonical forms η^i and connection forms $\eta^i_j = -\eta^i_j$. If $f : M \to \mathbb{R}^{n+1}$ is an affine isometric embedding, then an affine framing along $f(M)$, as constructed above, will correspond via f_* to an orthonormal frame on M, i.e., a section of B. The two framings can be regarded as lifts of M into B and into $\mathcal{F}$, and in this way we obtain an n-dimensional submanifold of $B \times \mathcal{F}$ along which the canonical forms η^i and ω^i coincide. (Consequently, the Levi-Civita connection forms η^i_j and τ^i_j must also coincide.) Thus, these 'graphs' will be integral submanifolds of the Pfaffian system

$$I = \{\omega^0, \omega^0_0, \omega^0_i - \omega^i, \eta^i - \omega^i, \eta^i_j - \tau^i_j\}$$

on $B\times\mathcal{F}$. Conversely, any n-dimensional integral submanifold of this system, on which the n-form $\omega^1\wedge\cdots\wedge\omega^n$ is nonzero at each point, gives the 'graph'—in terms of an identification of orthonormal frames—of an affine isometric embedding of M.

Once one has such an integral submanifold, others can be obtained by simultaneously rotating the frames in question; of course, this does not change the underlying embedding f, but it does show how the group $O(n)$ acts simultaneously in B and $\mathcal{F}$ to give automorphisms of the system I. The action of $O(n)$ generates a foliation of $B\times\mathcal{F}$ with $n(n-1)/2$-dimensional leaves; one refers to the directions along these leaves as *Cauchy characteristic directions.*

Let $\mathcal{I}_0$ be the differential ideal generated by I. To complete a set of generators for $\mathcal{I}_0$, we need to add the exterior derivatives:

$$\left.\begin{aligned} d\omega^0 &\equiv 0 \\ d(\omega^0_i - \omega^i) &\equiv 2\sigma^i_j \wedge \omega^j \\ d(\omega^0_0) &\equiv \omega^i_0 \wedge \omega^i \\ d(\eta^i - \omega^i) &\equiv \sigma^i_j \wedge \omega^j \\ 2d(\psi^i_j - \tau^i_j) &\equiv 2\sigma^i_k \wedge \sigma^k_j + \omega^i_0 \wedge \omega^j - \omega^j_0 \wedge \omega^i + R^i_{jkl}\omega^k \wedge \omega^l \end{aligned}\right\} \quad \text{mod } I. \tag{8}$$

Our main tool for the study of $\mathcal{I}_0$ and its prolongations will be the characteristic variety. Given an integral n-plane E of $\mathcal{I}_0$—i.e., an n-dimensional subspace of the tangent space at some point of $B\times\mathcal{F}$, to which all the forms in $\mathcal{I}_0$ restrict to be zero—a hyperplane in E is characteristic if E is not the only integral n-plane containing that hyperplane. The set of characteristic hyperplanes forms an algebraic variety $\Xi_E \subset \mathbb{P}(E^*)$, whose equations may be complexified to define the complex characteristic variety $\Xi_{\mathbb{C},E}$. We will only be concerned with integral n-planes which satisfy the *independence condition*

$$\omega^1 \wedge \cdots \wedge \omega^n|_E \neq 0. \tag{9}$$

Then any $\xi \in E^*$ can be expressed as $\xi_i\omega^i$, and we can use the ξ_i as homogeneous coordinates on $\mathbb{P}(E^*)$.

(If one strictly adheres to this definition, the fact that, in our case, one can always obtain a new integral plane by adding a Cauchy characteristic direction to E means that all hyperplanes are characteristic. To circumvent this, one can either work with a differential system defined on the quotient of $B \times \mathcal{F}$ by the $O(n)$ action, or one can adjoin the forms τ^i_j, which form a coframing along the Cauchy characteristic leaves, to the independence condition. In the latter case, one wants E to be $n(n+1)/2$-dimensional, but the points of Ξ_E all lie in the subspace spanned by the ω^i (see [Br2], §V.2). Since the same equations for Ξ_E are obtained when one simply ignores the forms τ^i_j in the calculation, this is what we will do.)

Proposition 2.1. *Let E be an integral n-plane of $\mathcal{I}_0$ satisfying (9). Then $\Xi_{\mathbb{C},E}$ is contained in the quadric defined by*

$$\xi_1^2 + \cdots + \xi_n^2 = 0. \tag{10}$$

Proof. As in (5), (6), $\sigma^i_j|_E = s_{ijk}\omega^k$ and $\omega^i_0|_E = 2L_{ij}\omega^j$. The E is annihilated by

$$\tilde\sigma^i_j := \sigma^i_j - A_{ijk}\omega^k$$
$$\tilde\omega^i_0 := \omega^i_0 - B_{ij}\omega^j.$$

The system 2-forms can be rewritten at the basepoint of E as

$$\tilde\omega^i_0 \wedge \omega^i, \qquad \tilde\sigma^i_j \wedge \omega^j, \qquad 2\tilde\sigma^i_k \wedge \tilde\sigma^k_j + \tilde\omega^i_0 \wedge \omega^j - \tilde\omega^j_0 \wedge \omega^i.$$

Let $\xi = \xi_i\omega^i$. Any integral element containing the hyperplane $\xi^\perp \subset E$ must be annihilated by the forms in I and by the forms

$$\begin{aligned} \theta_{kl} &= \xi_k\tilde\omega^l_0 - \xi_l\tilde\omega^k_0 \\ \theta^i_{kl} &= \xi_k\tilde\sigma^i_l - \xi_l\tilde\sigma^i_k \\ \theta^{ij}_{kl} &= \xi_k\tilde\omega^i_0\delta^j_l - \xi_l\tilde\omega^i_0\delta^j_k - \xi_k\tilde\omega^j_0\delta^i_l + \xi_l\tilde\omega^j_0\delta^i_k. \end{aligned} \tag{11}$$

These are obtained by wedging the above 2-forms with ξ and factoring out $\omega^k \wedge \omega^l$. We will show that, if (10) does not hold, the forms θ_{kl}, θ^i_{kl} and θ^{ij}_{kl} have the same span as the $\tilde\sigma^i_j$ and $\tilde\omega^i_0$. This will imply that E is the only integral element containing the hyperplane.

Let $\xi \cdot \xi = \xi_1^2 + \cdots + \xi_n^2$. Notice that

$$\sum_k \xi_k\theta^i_{kl} = \xi \cdot \xi\tilde\sigma^i_j - \xi_l\sum_k \xi_k\tilde\sigma^i_k$$

and

$$\sum_k \theta^k_{kl} = \sum_k \xi_k\tilde\sigma^k_l = \sum_k \xi_k\tilde\sigma^l_k.$$

Thus,

$$\sum_k \xi_k\theta^i_{kl} + \xi_l\theta^k_{ki} = (\xi\cdot\xi)\tilde\sigma^i_j,$$

which shows that the forms θ^i_{kl} have the same span as the $\tilde\sigma^i_j$.

It remains to obtain the n forms $\tilde\omega^i_0$ as combinations of the forms in (11). Disable the summation convention, and set $l = j$ in the third set of forms in (11). If $i \neq j$,

$$\sum_k \xi_k \theta^{ij}_{kj} = (\xi\cdot\xi - \xi_j^2)\tilde\omega^i_0 + \xi_i\xi_j\tilde\omega^j_0. \tag{12}$$

If (10) does not hold, there must be a j such that

$$\xi\cdot\xi - \xi_j^2 \neq 0. \tag{13}$$

Then (12) gives $n-1$ independent forms. For this j, either $\xi_j = 0$ or $\xi_j \neq 0$. If $\xi_j = 0$, then $\xi_i \neq 0$ for some $i \neq j$, and θ_{ij} is independent of the forms in (12). If $\xi_j \neq 0$, the form

$$\theta^{ij}_{ij} = \xi_i\tilde\omega^i_0 + \xi_j\tilde\omega^j_0$$

will be independent of those in (12) unless $\xi\cdot\xi - \xi_i^2 - \xi_j^2 = 0$. But if that happens for all $i \neq j$, then, assuming $n > 2$, we have $\xi\cdot\xi - \xi_j^2 = 0$ by addition, contradicting (13). Finally, if $n = 2$, it is easy to see that, when $\xi\cdot\xi \neq 0$, θ_{12} and θ^{12}_{12} have the same span as $\tilde\omega^1_0$, $\tilde\omega^2_0$. □

Corollary 2.2. *The affine isometric embedding system is elliptic for all n.*

The situation is very different for Euclidean isometric embedding. There, in the determined case, the system is never elliptic for $n \geq 3$ (see [Br1]).

3. The case $n = 3$ and the first prolongation

The prolongation of an exterior differential system with independence condition is obtained by restricting the canonical contact system to the space of integral elements. On any integral n-plane of $\mathcal{I}_0$ satisfying (9), the vanishing of the 2-forms (8) imply (5) and (6) for some s_{ijk} and L_{ij} satisfying the usual conditions. As well, the exterior derivatives of $\eta^i_j - \tau^i_j$ give the *Gauss equations*:

$$s_{ilm}s_{mjk} - s_{ikm}s_{mjl} + \delta_{il}L_{jk} - \delta_{ik}L_{jl} + \delta_{jk}L_{il} - \delta_{jl}L_{ik} = R_{ijkl}. \tag{14}$$

Tracing (14) gives the equation for the Ricci tensor of M:

$$s_{ijm}s_{kjm} + (2-n)L_{ik} - \delta_{ik}L_{jj} = R_{ik}, \tag{15}$$

which is equivalent to the Gauss equations when $n = 3$. So, when $n = 3$, the integral 3-planes of $\mathcal{I}_0$ can be parameterized by letting L_{ij} be defined by (15), with no restrictions on the s_{ijk}. (However, when $n > 3$, there are always quadratic restrictions on s_{ijk} involving the Weyl tensor of M.)

For $n = 3$, our prolongation $\mathcal{I}_1$ will be defined on $B \times \mathcal{F} \times S^3_0 V$. (In what follows, V stands for $\mathbb{R}^3$ and $S^k_0 V$ for the irreducible $SO(3)$ module of traceless symmetric k-forms on $\mathbb{R}^3$.) The one-forms of $\mathcal{I}_1$ will be those of $\mathcal{I}_0$ with the addition of

$$\begin{aligned} \tilde{\sigma}^i_j &:= \sigma^i_j - s_{ijk}\omega^k \\ \tilde{\omega}^i_0 &:= \omega^i_0 - 2L_{ij}\omega^j, \end{aligned} \tag{16}$$

where L_{ij} is defined by solving (15):

$$L_{ij} = s_{ikl}s_{jkl} - \frac{1}{4}|s|^2\delta_{ij} - E_{ij},$$

where $E_{ij} = R_{ij} - R/4\,\delta_{ij}$ is the Einstein tensor of M and $|s|^2 = s_{ijk}s_{ijk}$.

Let $\alpha_{ijk} = ds_{ijk} - s_{ljk}\tau^l_i - s_{ilk}\tau^l_j - s_{ijl}\tau^l_k$ and let $\alpha_{ij} = \frac{1}{2}(s_{ikl}\alpha_{jkl} + s_{jkl}\alpha_{ikl})$; note that α_{ijk} and α_{ij} are S^3_0V- and S^2_0V-valued 1-forms, respectively. Then the 2-forms of $\mathcal{I}_1$ are

$$\left.\begin{aligned} -d\tilde{\sigma}^i_j &\equiv \alpha_{ijk}\wedge\omega^k + L_{ik}\omega^k\wedge\omega^j + L_{jk}\omega^k\wedge\omega^i \\ -d\tilde{\omega}^i_0 &\equiv 2(dL_{ij} - L_{lj}\tau^l_i - L_{il}\tau^l_i)\wedge\omega^j + 2\sigma^i_j\wedge L_{jl}\omega^l \\ &\equiv 4\alpha_{ij}\wedge\omega^j - \alpha_{kk}\wedge\omega^i + 2E_{ijk}\omega^j\wedge\omega^k + 2s_{ijk}L_{jl}\omega^k\wedge\omega^l \end{aligned}\right\} \begin{matrix}\text{mod}\\ \text{1-forms,}\end{matrix} \tag{17}$$

where E_{ijk} are components of the covariant derivatives of the Einstein tensor with respect to the Levi-Civita connection on M.

We are now in a position to compute the characteristic variety of $\mathcal{I}_1$.

Proposition 3.1. *Let E be an integral 3-plane of $\mathcal{I}_1$ satisfying the independence condition (9). Then $\Xi_{\mathbb{C},E}$ is defined, in the usual homogeneous coordinates, by*

$$\xi\cdot\xi = 0, \qquad \sum_{i,j,k} s_{ijk}\xi_i\xi_j\xi_k = 0 \tag{18}$$

and

$$\sum_{k,l}(\xi_i s_{jkl} - \xi_j s_{ikl})\xi_k\xi_l = 0 \tag{19}$$

for all i and j.

Proof. Our computation depends only on the structure of $\mathcal{I}_1$ at the basepoint of E. Suppose $\alpha_{ijk}|_E = t_{ijkl}\omega^l$; then define new S^3_0V- and S^2_0V-valued 1-forms $\tilde{\alpha}_{ijk} = \alpha_{ijk} - t_{ijkl}\omega^l$ and $\tilde{\alpha}_{ij} = \frac{1}{2}(s_{ikl}\tilde{\alpha}_{jkl} + s_{jkl}\tilde{\alpha}_{ikl})$ annihilating E. The 2-forms of $\mathcal{I}_1$ in this basis are

$$\tilde{\alpha}_{ijk}\wedge\omega^k, \qquad 4\tilde{\alpha}_{ij}\wedge\omega^j - \tilde{\alpha}_{kk}\wedge\omega^i.$$

If $\xi \in E^*$, any integral 3-plane containing $\xi^\perp$ and satisfying the independence condition must be annihilated by the 1-forms of $\mathcal{I}_1$ and by

$$\tilde{\alpha}_{ijk}\xi_l - \tilde{\alpha}_{ijl}\xi_k, \qquad 4(\tilde{\alpha}_{ij}\xi_k - \tilde{\alpha}_{ik}\xi_j) - \tilde{\alpha}_{ll}(\delta_{ij}\xi_k - \delta_{ik}\xi_j).$$

The way in which these annihilators are linear combinations of the annihilators $\tilde{\alpha}_{ijk}$ of E defines the symbol map σ_ξ. In this case, $\sigma_\xi : S^3_0 V \to (S^2_0 V \otimes \Lambda^2 V) \oplus (V \otimes \Lambda^2 V)$ is given by

$$\sigma_\xi(a_{ijk}) = a_{ijk}\xi_l - a_{ijl}\xi_k \otimes 4(b_{ij}\xi_k - b_{ik}\xi_j) - b_{ll}(\delta_{ij}\xi_k - \delta_{ik}\xi_j), \qquad (20)$$

where $b_{ij} = \frac{1}{2}(s_{ikl}a_{jkl} + s_{jkl}a_{ikl})$. The hyperplane $\xi^\perp$ is characteristic when σ_ξ fails to be injective.

Suppose a_{ijk} is in the kernel of σ_ξ; then setting the first factor in (20) equal to zero shows that $a_{ijk} = \lambda \xi_i \xi_j \xi_k$. (In order for this to be in $S^3_0 V$, we must have $\xi \cdot \xi = 0$. In general, the characteristic variety of the prolongation will be contained, with appropriate identifications, in the original characteristic variety; see [Br2], §V.3.) Thus, the kernel is at most one-dimensional. With $\lambda = 1$, $b_{ij} = \frac{1}{2}(c_i\xi_j + c_j\xi_i)$, where $c_i = s_{ikl}\xi_k\xi_l$. Then setting the other factor of (20) equal to zero gives

$$2(\xi_i\xi_k c_j - \xi_i\xi_j c_k) - \xi_l c_l(\delta_{ij}\xi_k - \delta_{ik}\xi_j) = 0. \qquad (21)$$

Contraction with ξ_j, and using $\xi \cdot \xi = 0$, gives $\xi_i\xi_k c_j \xi_j = 0$ for all i and k. This implies $c_j\xi_j = 0$, the second equation in (18). Now (21) becomes

$$\xi_i(\xi_j c_k - \xi_k c_j) = 0$$

for all i, j, k. Since $\xi \neq 0$, this implies (19).□

Corollary 3.2. *At points where the Pick form $s_{ijk} \neq 0$, $\Xi_{\mathbb{C},E}$ consists of the double points of the intersection of the curves $\xi \cdot \xi = 0$ and $s_{ijk}\xi_i\xi_j\xi_k = 0$.*

Proof. The homogeneous coordinates for the tangent line to $\xi \cdot \xi = 0$ at the point ξ are just $[\xi_1, \xi_2, \xi_3]$. But (19) implies that the coordinates $[c_1, c_2, c_3]$ of the tangent line to $s_{ijk}\xi_i\xi_j\xi_k = 0$ at ξ are the same as $[\xi_1, \xi_2, \xi_3]$ up to scalar multiple.□

Proof of Theorem 1. Let $\mathcal{C} \subset S^3_0 V$ be the open subset of nonzero cubics whose intersection with the isotropic curve $\xi \cdot \xi = 0$ has no double points. Then the restriction of $\mathcal{I}_1$ to $B \times \mathcal{F} \times \mathcal{C}$ has empty characteristic variety at each point. Then by Thm. V.3.12 in [Br2], there is a suitable q such that each connected integral manifold of $\mathcal{I}_1$ is determined by its q-jet at one point. □

4. The tableau and its prolongations

The *tableau* of a linear Pfaffian system gives a pointwise, linear-algebraic way of obtaining information about the space of solutions. If $\theta^1, \cdots, \theta^s$ is a basis for the 1-forms of the system, with independence condition $\omega^1 \wedge \cdots \wedge \omega^n \neq 0$, and

$$d\theta^a \equiv \pi^a_i \wedge \omega^i \mod \theta^1, \cdots, \theta^s$$

then the tableau A is the subspace of $\mathbb{R}^s \otimes \mathbb{R}^n$ cut out by any linear relations satisfied by the forms π^a_i. If a certain number of the θ's span the first derived system, then the corresponding π^a_i's may be taken to be zero.

Since the 1-forms of $\mathcal{I}_0$ are automatically in the first derived system of the prolongation $\mathcal{I}_1$, and the remaining 1-forms are given by (16), the tableau of $\mathcal{I}_1$ lies in $W \otimes V$ for $W = S^2_0 V \oplus V$. Then (17) shows that the tableau is

$$A = \{a \oplus \varphi(a) \in (S^2_0 V \otimes V) \oplus (V \otimes V) | a \in S^3_0 V\},$$

where $\varphi : S^3_0 V \to S^2 V$ is the linear map defined component-wise by

$$b_{ij} = 2(s_{ikl}a_{jkl} + s_{jkl}a_{ikl}) - s_{klm}a_{klm}\delta_{ij}.$$

By definition, the prolongations of the tableau are $A^{(1)} = A \otimes V \cap W \otimes S^2 V$, $A^{(2)} = A^{(1)} \otimes V \cap W \otimes S^3 V$, etc. In our case,

$$A^{(1)} = \{a' \oplus \varphi'(a') | a' \in S^4_0 V, \varphi'(a') \in S^3 V\}$$

and

$$A^{(2)} = \{a'' \oplus \varphi''(a'') | a'' \in S^5_0 V, \varphi''(a'') \in S^4 V\},$$

where $\varphi' = \varphi \otimes \mathrm{id}$ and $\varphi'' = \varphi' \otimes \mathrm{id}$.

It is clear that $A^{(2)}$ is isomorphic to the subspace of $S^5_0 V$ defined by the equation $b_{ijkl} - b_{ikjl}$, where $b = \varphi''(a)$. Thus, $A^{(2)}$ is the kernel of a linear map $\psi : S^5_0 V \to V \otimes \Lambda^2 V \otimes V \cong V \otimes V \otimes V$. We can write this map in component form as

$$c_{ilm} = \epsilon_{mjk}(2s_{ipq}a_{jpqkl} + 2s_{jpq}a_{ipqkl} - s_{pqr}a_{pqrkl}\delta_{ij}), \tag{22}$$

where ϵ_{mjk} is the totally skew-symmetric permutation symbol. (Note that the first contraction on the right is automatically zero.)

Suppose $\psi(a'') = 0$. Then the contraction $c_{ilm}\epsilon_{iln} = 0$ gives

$$s_{pqr}a_{pqrnm} = 0, \tag{23}$$

and then by (22), $\epsilon_{mjk}s_{jpq}a_{ipqkl} = 0$, or

$$s_{jpq}a_{kpqil} - s_{kpq}a_{jpqil} = 0. \tag{24}$$

Since this equation implies (23) by tracing on j and l, (24) is the defining equation for $A^{(2)}$.

The left-hand side of (24) can be interpreted as a bilinear map $\Phi : S^3_0 V \otimes S^5_0 V \to \Lambda^2 V \otimes S^2_0 V \cong V \otimes S^2_0 V$ which is $SO(3)$-equivariant. By the Clebsch-Gordan formula,

$$S^3_0 V \otimes S^5_0 V \cong S^2_0 V \oplus S^3_0 V \oplus \cdots \oplus S^8_0 V$$

as $SO(3)$ modules, whilst $V \otimes S_0^2 V \cong V \oplus S_0^2 V \oplus S_0^3 V$. Thus the image of Φ must be an $SO(3)$ module in $S_0^2 V \oplus S_0^3 V$. It is easy to verify that compositions with the projections into $S_0^2 V$ and $S_0^3 V$ are not automatically zero. So, our map Φ is essentially, up to normalizations, the sum of two Clebsch-Gordan homomorphisms on $S_0^3 V \otimes S_0^5 V$.

Proof of Theorem 2. Setting $\sigma = s_{ijk} x^i x^j x^k$ and $\alpha = a_{ijklm} x^i x^j x^k x^l x^m$ shows that Φ is precisely the map defined in (1). Hence when the Pick form is $SO(3)$-generic, $A^{(2)} = 0$. But for a linear Pfaffian system with tableau A, the space of integral elements satisfying the independence condition has the structure of an affine bundle with fibre isomorphic to $A^{(1)}$. In our case, A is the tableau of $\mathcal{I}_1$, $A^{(1)}$ the tableau for $\mathcal{I}_2$, and $A^{(2)} = 0$ implies that the prolongation $\mathcal{I}_2$ has at most a unique integral element at each point.[1]

Just as the components s_{ijk} of the Pick form were added as new variables to define $\mathcal{I}_1$, the components s_{ijkl} of the (symmetrized) covariant derivatives are the only new variables added in defining $\mathcal{I}_2$. Hence, the value of the Pick form σ and of $\nabla\sigma$ on M uniquely determine a point in the manifold on which $\mathcal{I}_2$ is defined, and our genericity condition implies that there can be at most a unique integral manifold of $\mathcal{I}_2$ through that point. $\square$

5. Special Hypersurfaces in $\mathbb{R}^4$

For an involutive EDS, the value of the last nonzero Cartan character s_l determines the size of the space of integral manifolds. For example, if $s_l = k$, local integral manifolds can be constructed (using the Cartan-Kähler theorem) by choosing k arbitrary functions of l variables. As well, the characteristic variety will have dimension $l - 1$. Thus, if we hope that the affine isometric embeddings of a given Riemannian manifold M depend on more than just a choice of constants, we had better look among those hypersurfaces for which the characteristic variety for the system $\mathcal{I}_1$ is non-empty.

As shown in §3, this means that, at every point of M, the Pick curve must intersect the isotropic curve in double points. At any point of the hypersurface, orthonormal frames can be chosen so that $e_1 \pm ie_2$ are the double points, in which case the components of the Pick form satisfy

$$\begin{array}{lll} s_{123} = 0 & & \\ s_{113} = a, & s_{223} = a, & s_{333} = -2a \\ s_{122} = -b, & s_{133} = 4b, & s_{111} = -3b \\ s_{112} = -c, & s_{233} = 4c, & s_{222} = -3c \end{array} \tag{25}$$

Moreover, if $b^2 + c^2 \neq 0$—equivalently, if the double points never degenerate to triple points—the frames may be chosen smoothly along M. This leads to

[1] Integral elements for $\mathcal{I}_1$ exist, for example, at each point where the Clebsch-Gordan homomorphism $S_0^3 V \otimes S_0^4 V \to S_0^2 V$, with the Pick form in the first slot, is surjective. Our genericity assumption implies the latter condition, but does not guarantee that torsion is absorbable at the level of $\mathcal{I}_2$.

the first of the EDS's we use in this section; for the second system, we will assume b and c are identically zero.

Although our examples of nonrigidity, which arise as a subcase of $b = c = 0$, depend in fact only on a choice of constants, we feel that these nevertheless illustrate the fundamental role of the Pick form—even when it is nonzero!—in affine differential geometry, and the utility of the characteristic variety in guiding the EDS machinery.

An affine orthonormal frame along a hypersurface in $\mathbb{R}^4$, with Pick form as in (25), corresponds to an integral 3-fold of the Pfaffian system

$$J = \begin{cases} \omega_i^0 - \omega^i \\ \sigma_1^1 + 3b\omega^1 + c\omega^2 - a\omega^3 \\ \sigma_2^2 + b\omega^1 + 3c\omega^2 - a\omega^3 \\ \sigma_3^3 - 4b\omega^1 - 4c\omega^2 + 2a\omega^3 \\ \sigma_2^1 + c\omega^1 + b\omega^2 \\ \sigma_3^1 - a\omega^1 - 4b\omega^3 \\ \sigma_3^2 - a\omega^2 - 4c\omega^3 \end{cases} \tag{26}$$

on $\mathcal{F} \times \mathbb{R}^3$, where we add a, b, c as new variables. The 1-forms $\omega_i^0 - \omega^i$, $i = 1, 2, 3$, are closed modulo J; these then span the first derived system of J. The exterior derivatives of the remaining 1-forms are

$$\begin{cases} (3\beta_1 - \omega_0^1) \wedge \omega^1 + \beta_2 \wedge \omega^2 - (5\alpha_1 + \alpha_0) \wedge \omega^3 \\ \beta_1 \wedge \omega^1 + (3\beta_2 - \omega_0^2) \wedge \omega^2 + (5\alpha_1 - \alpha_0) \wedge \omega^3 \\ -4\beta_1 \wedge \omega^1 - 4\beta_2 \wedge \omega^2 + (2\alpha_0 - \omega_0^3) \wedge \omega^3 \\ (\beta_2 - \frac{1}{2}\omega_0^2) \wedge \omega^1 + (\beta_1 - \frac{1}{2}\omega_0^1) \wedge \omega^2 - 5\alpha_2 \wedge \omega^3 \\ (-\alpha_0 - 5\alpha_1 - \frac{1}{2}\omega_0^3) \wedge \omega^1 - 5\alpha_2 \wedge \omega^2 + (-4\beta_1 - \frac{1}{2}\omega_0^1) \wedge \omega^3 \\ -5\alpha_2 \wedge \omega^1 + (-\alpha_0 - 5\alpha_1 - \frac{1}{2}\omega_0^3) \wedge \omega^2 + (-4\beta_2 - \frac{1}{2}\omega_0^2) \wedge \omega^3, \end{cases} \tag{27}$$

where we have defined

$$\begin{aligned} \alpha_0 &:= da + 6(b\tau_3^1 + c\tau_3^2) \\ \alpha_1 &:= b\tau_3^1 - c\tau_3^2 \\ \alpha_2 &:= c\tau_3^1 + b\tau_3^2 \\ \beta_1 &:= db + c\tau_2^1 - a\tau_3^1 \\ \beta_2 &:= dc - b\tau_2^1 - a\tau_3^2 \end{aligned}$$

Let $\mathcal{J}_0$ be the differential ideal generated by J. For an integral 3-plane E of this system, satisfying $\omega^1 \wedge \omega^2 \wedge \omega^3|_E \neq 0$, the characteristic variety consists of the single point $\xi = \omega^3$. The Cartan characters of $\mathcal{J}_0$ are $s_1 = 6$, $s_2 = 2$, $s_3 = 0$, but the space of integral elements at each point has dimension $9 < s_1 + 2s_2 + 3s_3$ (see next paragraph). The system fails to be involutive.

We may define the prolongation $\mathcal{J}_1$ by introducing the 9 variables $L_{ij} = L_{ji}$ and e, f, g, and defining the 1-forms

$$\begin{aligned}
\theta_i &:= L_{ij}\omega^j - \frac{1}{2}\omega_0^i \\
\bar{\alpha_0} &:= \alpha_0 + (L_{13} + 10e)\omega^1 + (L_{23} + 10g)\omega^2 + (L_{33} - 4f + \frac{3}{2}(L_{11} + L_{22}))\omega^3 \\
\bar{\alpha_1} &:= \alpha_1 + e\omega^1 - g\omega^2 + \frac{1}{2}(L_{22} - L_{11})\omega^3 \\
\bar{\alpha_2} &:= \alpha_2 + g\omega^1 + e\omega^2 - L_{12}\omega^3 \\
\bar{\beta_1} &:= \beta_1 + (L_{22} - f)\omega^1 - L_{12}\omega^2 - (5e + L_{13})\omega^3 \\
\bar{\beta_2} &:= \beta_2 + (L_{11} - f)\omega^2 - L_{12}\omega^1 - (5g + L_{23})\omega^3.
\end{aligned}$$

Since only six 1-forms in J remain after we exclude the first derived system, the tableau A of $\mathcal{J}_0$ is contained in $W \otimes V$ where $\dim W = 6$. Since $s_1 = \dim W$, it follows that, in the sequence of linear maps

$$A^{(2)} \hookrightarrow A^{(1)} \otimes V \longrightarrow A \otimes \Lambda^2 V \longrightarrow W \otimes \Lambda^3 V,$$

the rightmost map surjects. (Note that the sequence is exact at the second term, by the definition of $A^{(2)}$.) Since $\dim A = 8$—corresponding the the forms $\omega_0^1, \omega_0^2, \omega_0^3$ and $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2$—and $\dim A^{(1)} = 9$—corresponding to the parameters L_{ij} and e, f, g—and $\dim A^{(2)} = 9$, this sequence is also exact at the third term. It follows that, since the torsion of $\mathcal{J}_1$ lies in the kernel of the rightmost map, that torsion is always absorbable by a change of coframe. That is, there will be forms $DL_{ij} = dL_{ij} + \cdots$, $De = de + \cdots$, etc., such that the prolongation 2-forms are

$$\begin{array}{l}
DL_{ij} \wedge \omega^j \\
10De \wedge \omega^1 + 10Dg \wedge \omega^2 - (4Df + \frac{3}{2}(DL_{11} + DL_{22})) \wedge \omega^3 \\
De \wedge \omega^1 - Dg \wedge \omega^2 + \frac{1}{2}(DL_{22} - DL_{11}) \wedge \omega^3 \\
Dg \wedge \omega^1 + De \wedge \omega^2 - DL_{12} \wedge \omega^3 \\
(DL_{11} + DL_{22} - Df) \wedge \omega^1 - 5De \wedge \omega^3 \\
(DL_{11} + DL_{22} - Df) \wedge \omega^2 - 5Dg \wedge \omega^3.
\end{array} \tag{28}$$

(We have subtracted some multiples of the forms $DL_{ij} \wedge \omega^j$ from the others in order to simplify the tableau.)

If we test $\mathcal{J}_1$ for involutivity, we find that $s_1 = 8$, $s_2 = 1$, $s_3 = 0$; and since $\dim A^{(2)} = 9 < s_1 + 2s_2 + 3s_3$, the test fails again. On the other hand, the fact that $\dim A^{(3)} = 9$ indicates, by cohomological considerations discussed in the appendix, that there are 2-forms outside the ideal $\mathcal{J}_1$ but which vanish on all integral 3-planes. In fact, it is clear from the last two rows of (28) that $\phi := Df - DL_{11} - DL_{22}$ must be a multiple of ω^3. When we add the 2-form $\phi \wedge \omega^3$ to the ideal, $s_1 = 9$, $s_2 = 0$ and $s_3 = 0$. The system is now involutive.

Thus, special hypersurfaces of the type with $b^2 + c^2 \neq 0$ depend locally on a choice of nine arbitrary functions of one variable.

For hypersurfaces whose Pick curve always intersects the isotropic curve in triple points, we use a Pfaffian system $\tilde{J}$ on $\mathcal{F} \times \mathbb{R}$, whose forms are the same as (26),(27), except with $b = c = 0$. (We also assume $a \neq 0$.) The Cartan characters of $\tilde{\mathcal{J}}_0$ are $s_1 = 4$, $s_2 = 2$, $s_3 = 0$, and $\dim \tilde{A}^{(1)} = 5 < s_1 + 2s_2 + 3s_3$.

The forms for the first prolongation $\tilde{\mathcal{J}}_1$ are

$$\begin{aligned}
\theta_i &:= L_{ij}\omega^j - \frac{1}{2}\omega^i_0 \qquad \text{(but with } L_{12} = 0 \text{ and } L_{22} = L_{11}) \\
\bar{\alpha}_0 &:= da + L_{13}\omega^1 + L_{23}\omega^2 + (L_{33} - L_{11} - 4F)\omega^3 \\
\bar{\beta}_1 &:= -a\tau^1_3 - F\omega^1 - L_{13}\omega^3 \\
\bar{\beta}_2 &:= -a\tau^2_3 - F\omega^2 - L_{23}\omega^3.
\end{aligned}$$

(Here, we have changed to $F = f - L_{11} = f - L_{22}$.) The Cartan characters for $\tilde{\mathcal{J}}_1$ are $s_1 = 4$, $s_2 = 1$, $s_3 = 0$, while $\dim \tilde{A}^{(2)} = 4$; the system again fails to be involutive. However, by cohomological methods discussed in the appendix, we may detect that there is a 1-form not in $\tilde{\mathcal{J}}_1$ but which vanishes on all integral elements. In fact,

$$\left.\begin{aligned}
d(\theta_1 + \bar{\beta}_1) &\equiv (dL_{11} - dF - 2aL_{23}\omega^2 + \frac{3a^4 + 5F^2 - 2a^2L_{33}}{a}\omega^3) \wedge \omega^1 \\
d(\theta_2 + \bar{\beta}_2) &\equiv (dL_{11} - dF - 2aL_{13}\omega^1 + \frac{3a^4 + 5F^2 - 2a^2L_{33}}{a}\omega^3) \wedge \omega^2
\end{aligned}\right\} \text{mod 1-forms}$$

shows that the missing 1-form is

$$dL_{11} - dF - 2aL_{13}\omega^1 - 2aL_{23}\omega^2 + \frac{3a^4 + 5F^2 - 2a^2L_{33}}{a}\omega^3.$$

Once this 1-form is added to the system, the characters become $s_1 = 4$, $s_2 = 0$, $s_3 = 0$, and the new system is involutive. Thus, special hypersurfaces with $b = c = 0$ depend locally on a choice of four functions of one variable.

If we wish to show that in some cases there are essentially different affine isometric embeddings for a given metric, we have to start with the metric itself, and show that the embedding problem has more than one solution. Unfortunately, the preceding analysis of special hypersurfaces does not give us much intrinsic information about the metrics themselves. However, if we introduce the "ansatz"

$$L_{13} = L_{23} = 0, \tag{29}$$

the equations for the $b = c = 0$ hypersurfaces become much simpler. Namely, along these hypersurfaces $\tau^1_3 = -\dfrac{F}{a}\omega^1$ and $\tau^2_3 = -\dfrac{F}{a}\omega^2$, and so

$$d\omega^3 = -\tau^3_1 \wedge \omega^1 - \tau^3_2 \wedge \omega^2 = 0.$$

This means the 2-planes spanned by e_1, e_2 are tangent to a foliation of M. The metric restricts to the leaves to be $(\omega^1)^2 + (\omega^2)^2$, and the Lie derivative

$$\mathcal{L}_{e_3}\left((\omega^1)^2 + (\omega^2)^2\right) = -\frac{F}{a}\left((\omega^1)^2 + (\omega^2)^2\right)$$

shows that the restriction of the metric changes only by a conformal factor as one moves perpendicularly to the leaves. When we substitute (29) into the 1-forms of $\tilde{\mathcal{J}}_1$, we find that on an integral manifold, da and $d(L_{11} - F)$ must be multiples of ω^3—i.e., a and $L_{11} - F$ are constant along the leaves. As well, substituting (29) into the 2-form

$$d(\bar{\alpha}_0 - \theta_3) \equiv -5\left(dF + \frac{a^2 + F}{a}(L_{13}\omega^1 + L_{23}\omega^2)\right) \wedge \omega^3$$

shows that F is also constant along the leaves. Lastly,

$$d\tau^1_2 = \left(\frac{F^2}{a^2} - 2L_{11} - a^2\right)\omega^1 \wedge \omega^2$$

shows that the curvature of the metric along the leaves is constant on each leaf. Hence our metric on M is a warped product, over a one-dimensional base, of constant curvature surfaces.

We may ask now, if every such warped product has an affine isometric embedding of this type. To answer this, we will specialize the isometric embedding system $\mathcal{I}_0$ to the case where the Riemannian manifold M is of this type.

Suppose the metric on M is $dt^2 + f(t)^2 ds^2$, where ds^2 is a metric of constant Gauss curvature K_0 on a surface. Suppose θ^1, θ^2 is an orthonormal coframing along the surface, with connection form ρ and the usual structure equations

$$d\theta^1 = -\rho \wedge \theta^2, \quad d\theta^2 = \rho \wedge \theta^2, \quad d\rho = K_0 \theta^1 \wedge \theta^2.$$

Then $\eta^1 = f(t)\theta^1$, $\eta^2 = f(t)\theta^2$, $\eta^3 = dt$ forms an orthonormal coframing along M, with connection forms $\eta^1_2 = \rho$, $\eta^1_3 = f'(t)\theta^1$, $\eta^2_3 = f'(t)\theta^2$.

In addition to specializing the metric on M, we will also require the Pick form to have the special form (25) with $b = c = 0$. Adjoining a as a new variable, our EDS will be defined on $M \times \mathcal{F} \times \mathbb{R}$, and generated by the 1-forms

$$\begin{cases} \omega^0, \omega^0_i - \omega^i \\ \omega^1 - f(t)\theta^1, \omega^2 - f(t)\theta^2, \omega^3 - dt \\ \sigma^1_1 - a\omega^3, \sigma^2_2 - a\omega^3, \sigma^3_3 + 2a\omega^3 \\ \sigma^1_2, \sigma^1_3 - a\omega^1, \sigma^2_3 - a\omega^2 \\ \tau^1_2 - \rho, \tau^1_3 - f'(t)\theta^1, \tau^2_3 - f'(t)\theta^2 \end{cases}$$

The 2-forms of this system are

$$
\begin{aligned}
d(a\omega^3 - \sigma^1_1) &\equiv da \wedge \omega^3 + \omega^1_0 \wedge \omega^1 \\
d(a\omega^3 - \sigma^2_2) &\equiv da \wedge \omega^3 + \omega^2_0 \wedge \omega^2 \\
d(\sigma^3_3 + 2a\omega^3) &\equiv 2da \wedge \omega^3 - \omega^3_0 \wedge \omega^3 \\
d(-2\sigma^1_2) &\equiv \omega^1_0 \wedge \omega^2 + \omega^2_0 \wedge \omega^1 \\
d(a\omega^1 - \sigma^1_3) &\equiv da \wedge \omega^1 - 4a\frac{f'}{f}\omega^1 \wedge \omega^3 + \frac{1}{2}(\omega^1_0 \wedge \omega^3 + \omega^3_0 \wedge \omega^1) \\
d(a\omega^2 - \sigma^2_3) &\equiv da \wedge \omega^2 - 4a\frac{f'}{f}\omega^2 \wedge \omega^3 + \frac{1}{2}(\omega^2_0 \wedge \omega^3 + \omega^3_0 \wedge \omega^2) \\
d(\tau^1_2 - \rho) &\equiv \left(\left(\frac{f'}{f}\right)^2 - a^2 - \frac{K_0}{f^2}\right)\omega^1 \wedge \omega^2 - \frac{1}{2}\omega^1_0 \wedge \omega^2 + \frac{1}{2}\omega^2_0 \wedge \omega^1 \\
d(\tau^1_3 - f'\theta^1) &\equiv \left(\frac{f''}{f} + 3a^2\right)\omega^1 \wedge \omega^3 - \frac{1}{2}\omega^1_0 \wedge \omega^3 + \frac{1}{2}\omega^3_0 \wedge \omega^1 \\
d(\tau^2_3 - f'\theta^2) &\equiv \left(\frac{f''}{f} + 3a^2\right)\omega^2 \wedge \omega^3 - \frac{1}{2}\omega^2_0 \wedge \omega^3 + \frac{1}{2}\omega^3_0 \wedge \omega^2.
\end{aligned}
\tag{30}
$$

The vanishing of the first pair of forms in (30) implies that $da \wedge \omega^3 = 0$; this is another "missing form" we should add to the system. We replace the first three forms in (30) with

$$da \wedge \omega^3, \quad \omega^1_0 \wedge \omega^1, \quad \omega^2_0 \wedge \omega^2, \quad \omega^3_0 \wedge \omega^3.$$

When these vanish on an integral 3-plane which satisfies the independence condition (9),

$$\omega^1_0 = 2L_{11}\omega^1, \quad \omega^2_0 = 2L_{22}\omega^2, \quad \omega^3_0 = 2L_{33}\omega^3$$

for some L_{11}, L_{22}, L_{33}. Substituting these values in the middle three forms in (30) shows that $L_{11} = L_{22}$, and forces

$$da = (L_{11} - L_{33} - 4a\frac{f'}{f})\omega^3. \tag{31}$$

Finally, substituting into the last three forms in (30) shows that

$$2L_{11} = \left(\frac{f'}{f}\right)^2 - a^2 - \frac{K_0}{f^2}$$

and

$$L_{33} = 3a^2 + \frac{f''}{f} - L_{11}.$$

Since L_{11} and L_{33} are completely determined, there is a unique integral 3-plane at each point of $M \times \mathcal{F} \times \mathbb{R}$. It is easy to verify that this distribution satisfies the Frobenius condition.

When we substitute the values for L_{11} and L_{33} into (31), we find that a satisfies a Riccati equation:

$$\frac{da}{dt} = -4a^2 - 4a(\log f)' - (\log f)'' - \frac{K_0}{f(t)^2}. \tag{32}$$

Recall that, once the nonzero component a of the Pick form is specified, the affine isometric embedding of M into $\mathbb{R}^4$ is determined up to affine motion. However, choosing different initial values for a in the ODE (32) will generate different Pick forms, and hence generate local solutions to the affine isometric problem that are not equivalent under affine motion.

It is not difficult to write down choices of $f(t)$ and K_0 for which (32) has solutions defined for all $t \in \mathbb{R}$. However, (32) also has solutions that give affine isometric embeddings for a compact M^3. Specifically, take $K_0 = 1$, and suppose $f(t)$ is positive on $(0, \pi)$ and zero at the endpoints. In order for the metric to be smooth at the endpoints, $f'(0) = 1$, $f'(\pi) = -1$, and f must be a C^∞ odd function of t and of $t - \pi$. The example we will use is

$$f(t) = \sin(t)\left(1 + (\alpha \cos t + \beta)\sin^2 t\right), \tag{33}$$

where we assume (α, β) lies strictly above the parametrized curve $\alpha = -\cos t/\sin^4 t$, $\beta = (2\cos^2 t - \sin^2 t)/\sin^4 t$ to get the second factor in (33) to be positive.

In order for the Pick form $a\,dt(2dt^2 - 3((\omega^1)^2 + (\omega^2)^2))$ to extend to be a smooth cubic form at the endpoints, $a(t)$ must also be a C^∞ odd function of t and $t - \pi$. Because f is odd, this is automatic provided $a(t)$ is bounded near the singular points $t = 0$ and $t = \pi$. With our choice of f, the only singular term in the right-hand side of (32) is $-4a(\log f)'$; in fact the term $-(\log f)'' - 1/f^2$ vanishes to first order at the endpoints. For $t \in (0, \pi)$, the solution curves for (32) will be the same as those of the system

$$\begin{aligned} dt/d\tau &= f(t) \\ da/d\tau &= -4a^2 f(t) - 4af'(t) - f''(t) + \frac{f'(t)^2 - 1}{f(t)}. \end{aligned}$$

This has saddle points at $a = 0$ and $t = 0$ or π. The unstable manifold of the first point and the stable manifold of the second point represent solutions to (32) that are bounded near $t = 0$ and near $t = \pi$ respectively. Sometimes these are one and the same curve; e.g. for $\alpha = 0$ and $\beta = 0$, $a(t) = 0$ is the only bounded solution, giving the isometric embedding of the standard metric on S^3 as a quadric in $\mathbb{R}^4$. However, this is not the only choice of (α, β) for which there exists a bounded solution to (32) on $[0, \pi]$.

When $\alpha = 0$ and $\beta = -.1$, the term $-(\log f)'' - 1/f^2$ is strictly negative for $t \in (0, \pi)$. This implies that the solution that is bounded near $t = 0$ tends to $-\infty$ as $t \nearrow \pi$, and the solution that is bounded near $t = \pi$ tends to $+\infty$ as $t \searrow 0$. Likewise, when $\alpha = 0$ and $\beta = 1$, or when $\alpha = \pm 1$ and $\beta = 2$, this term is strictly positive, and the behaviour of the two solutions is reversed. If we connect the first pair of parameter values to either of the other two pairs by a continuous curve within the set of admissable values, then there must be some point along the curve for which the two solutions coincide, and (32) has a bounded, smooth solution on $[0, \pi]$ which can be extended to a smooth function with the required symmetries.

Appendix: Saturation of linear Pfaffian systems

For the case of linear Pfaffian systems, we will derive certain results that relate the phenomenon of "missing" forms—that is, differential forms not contained in an EDS, but which vanish on all integral elements of that system—to the dimensions of the tableaux of the system. Formally, a Pfaffian system on M is a sub-bundle $I \subset T^*M$, with a local basis $\theta^1, \cdots, \theta^s$ for sections. Adding an independence condition—like $\omega^1 \wedge \cdots \wedge \omega^n \neq 0$ on integral n-manifolds—amounts to giving a pair of bundles $I \subset J \subset T^*M$ with rank $J/I = n$. (For these and other bundles we consider, we will assume constant rank.)

At points of M where there exist integral n-planes satisfying the independence condition, we can assume there are structure equations

$$d\theta^a \equiv \pi^a_i \wedge \omega^i \mod \theta^1, \cdots, \theta^s.$$

The forms π^a_i are not necessarily independent. If their linear dependencies at $x \in M$ are $B^{\rho i}_a(x)\pi^a_i \equiv 0 \mod \theta' s$, $1 \leq \rho \leq r$, then the tableau at x is the space $A \subset (J/I)_x \otimes I^*_x$ defined by $B^{\rho i}_a(x)(v_i \otimes w^a) = 0$. (We will use $W = I^*_x$ and $V = (J/I)_x$.)

For the differential ideal $\mathcal{I}$ generated by I, the $\theta^1, \cdots, \theta^s$ and the 2-forms $\pi^a_i \wedge \omega^i$ are algebraic generators. We are interested in the additional 1-forms and 2-forms which are linear in the π^a_i's and vanish on all integral n-planes satisfying the independence condition. Any such n-plane E has

$$\theta^a|_E = 0$$
$$\pi^a_i|_E = p^a_{ij}\omega^j$$

with $B^{\rho i}_a p^a_{ij} = 0$ and $p^a_{ij} = p^a_{ji}$—that is, p^a_{ij} are the components of an element of $A^{(1)} \subset W \otimes S^2V$.

The space of missing 1-forms is

$$\{\pi = c^i_a \pi^a_i \mid c^i_a p^a_{ij} = 0, \forall (p^a_{ij}) \in A^{(1)}\}$$

modulo A. Hence it is the kernel of the natural map

$$A^* \xrightarrow{\eta} (A^{(1)})^* \otimes V.$$

The dual of this space is isomorphic to the cokernel of the contraction map

$$A^{(1)} \otimes V^* \to A.$$

However, the dimension of the image here is, by definition, the contraction rank of $A^{(1)}$. So, we have

Lemma 1. *The space of missing 1-forms has dimension* $\dim A - \mathrm{rk}(A^{(1)})$.

The space of all 2-forms which are linear in the π_i^a's and vanish on the integral elements is

$$\{\Omega = c_{ak}^i \pi_i^a \wedge \omega^k \mid c_{ak}^i p_{ij}^a \omega^j \wedge \omega^k = 0, \forall (p_{ij}^a) \in A^{(1)}\}.$$

This space is the kernel of the composition

$$A^* \otimes V \xrightarrow{\eta \otimes \mathrm{id}} (A^{(1)})^* \otimes V \otimes V \xrightarrow{\text{wedge}} (A^{(1)})^* \otimes \Lambda^2 V. \tag{34}$$

We wish to 'mod out' by the 2-forms $\pi_i^a \wedge \omega^i$, which correspond to $c_{ak}^i = c_a \delta_k^i$, i.e., the image of

$$W^* \xrightarrow{\otimes \mathrm{id}} W^* \otimes V^* \otimes V \xrightarrow{\text{restrict}} A^* \otimes V. \tag{35}$$

Taking the duals of (34) and (35) gives a sequence of maps

$$A^{(1)} \otimes \Lambda^2 V^* \xrightarrow{\text{contract}} A \otimes V^* \xrightarrow{\text{contract}} W. \tag{36}$$

Note that the image of the first map is in the kernel of the second. The dual of the space of missing 2-forms is isomorphic to the cohomology at the middle term.

When $\dim V = 3$, picking a volume form on V gives isomorphisms $\Lambda^3 V^* \cong \mathbb{R}$ and $\Lambda^2 V^* \cong V$, and the first map in (36) can be replaced by

$$A^{(1)} \otimes V \hookrightarrow A \otimes V \otimes V \xrightarrow{\text{wedge}} A \otimes \Lambda^2 V.$$

By definition, the kernel of this map is $A^{(2)}$. On the other hand, the image of the second map in (36) has dimension $\mathrm{rk}(A)$. Putting these two facts together gives

Lemma 2. *When $n = 3$, the space of missing 2-forms has dimension*

$$3 \dim A - \mathrm{rk}(A) - (3 \dim A^{(1)} - \dim A^{(2)}).$$

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