Integrable geometric evolution equations for curves

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ABSTRACT. The vortex filament flow and planar filament flow are examples of evolution equations which commute with Euclidean isometries and are also integrable, in that they induce completely integrable PDE for curvature—the focusing nonlinear Schödinger equation and the mKdV equations, respectively. In this note we outline an approach for classifying integrable geometric evolution equations for planar curves, using necessary conditions derived by Mikhailov *et al*, based on generalized symmetries of arbitrarily high order. Here we give new examples of integrable third-order curve flows obtained by this classification, and discuss their conservation laws, recursion operators, and related flows for curves in $\mathbb{R}^3$.

Introduction

In this paper, we'll discuss approaches and results in identifying those geometric evolution equations for curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ which are completely integrable. Here, an evolution equation for a curve is *geometric* if the velocity is expressed purely in terms of objects invariant under Euclidean motions, i.e., the Frenet frame, the curvature and torsion of the curve, and finitely many of their arclength derivatives. We'll say such a flow is integrable if it induces a completely integrable system of PDE for curvature and torsion.

There are two examples which motivate this classification project. First, the *vortex filament flow* is an evolution equation for space curves which was introduced in 1906 by L. Da Rios (a student of Levi-Civita) as a model for the motion of one-dimensional vortex filaments in an incompressible fluid. Letting x be the arclength parameter for the curve γ, the flow is

(1) $$\gamma_t = \gamma' \times \gamma'',$$

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where the prime denotes derivative with respect to x. Note that the right-hand side equals the curvature κ times the binormal vector B of γ. In the 1970s, Hasimoto found that, at the level of curvature and torsion, (1) is equivalent, up to phase, to the focusing cubic nonlinear Schrödinger (NLS) equation, a well-known completely integrable PDE.

Later, Langer and Perline [**3**] translated the NLS hierarchy to a hierarchy of geometric flows which commute with (1). The next flow after (1) in this hierarchy restricts to give an evolution for planar curves called the *planar filament flow* [**5**]:

$$\gamma_t = \tfrac{1}{2}\kappa^2 T + \kappa' N, \tag{2}$$

where T, N are the unit tangent and normal, defined so that $T' = \kappa N$. Along with (1), this flow has the property that it preserves an arclength parameter.[1] Differentiating (2) twice with respect to arclength, and equating mixed partials, gives

$$\kappa_t = \kappa''' + \tfrac{3}{2}\kappa^2\kappa'.$$

So, the planar filament flow induces a completely integrable PDE for κ, the mKdV equation.

Which other curve flows, either planar or in three dimensions, are completely integrable in this sense? To begin to answer this, we have to sort out what is meant by complete integrability. The relevant features of the most well-known PDE that carry this designation—the KdV, mKdV, sine-Gordon, and NLS equations—fall into two main areas: solvability by inverse scattering, and Hamiltonian or H-integrability, for short. Inverse scattering requires finding a linear differential operator L on the line—with coefficients depending on the unknown(s) u in the PDE—for which the given PDE constitutes an isospectral flow. (Of course, L is part of the Lax pair for the PDE.) This sense of integrability is unsuitable for classification problems because, in most cases, L is found after the PDE has been identified. Furthermore the algebraic appearance of u or its derivatives in L, of importance in the construction of finite-gap and soliton solutions, is highly coordinate-dependent, whereas we would like to impose necessary conditions which are independent of coordinates. Meanwhile, H-integrability requires the existence of an infinite number of independent constants of motion (in the form of conservation laws) for the flow, involving arbitrarily high derivatives of u, which commute with respect to some Poisson structure. (We leave aside the issue of whether or not these constants of motion are in any sense complete.) The requirement of infinitely many local conservation laws

[1] In general, a geometric evolution equation $\gamma_t = W$ has this property if W' is orthogonal to T; the condition is the same for flows for curves in $\mathbb{R}^3$ (cf. [**3**], §3).

is independent of coordinates, and could be posed in terms of characteristic cohomology [**1**]. However, the classification detailed below is carried out under the stricter assumption—which in most cases implies H-integrability—that there exists a generalized symmetry of arbitrarily high rank for the given evolution equation. As explained below, this usually leads to a recursion operator allowing us to generate the requisite hierarchy of conservation laws.

In using this technique, we are following the work of Mikhailov, Shabat and Sokolov [**7**], hereinafter referred to as MSS, who listed (up to contact transformations) all potentially H-integrable evolution equations of various types (including all third-order scalar evolution equations, and a sub-class of scalar fifth-order equations). Here, we are considering only those PDE arising from geometric evolution equations. For this smaller set of candidate equations, our goal is to obtain more detailed information on complete integrability: discovering explicit recursion operators, determining the bi-Hamiltonian structure, and eventually deriving Lax pairs and Bäcklund transformations. Ideally, understanding the geometry of the flow should help in identifying these structures.

The rest of the paper is organized into the following sections: background material on generalized symmetries and canonical densities; classifying integrable flows for planar curves; recursion operators for some of the flows found; and a discussion of further results on related flows.

While the classification project described here is not yet finished, it would not have begun without the valuable assistance of Ian Anderson and Mark Fels of Utah State University. The author also wishes to thank the organizers, Joshua Leslie and Thierry Robart of Howard University, for the opportunity to participate in this workshop, and to thank Joel Langer, Peter Olver and David Singer for their comments and suggestions.

1. Generalized Symmetries and Canonical Densities

In this section we summarize the background material for the classification below; a complete exposition is available in [**8**]. Suppose we have an autonomous evolution equation of order m for scalar $u(x,t)$,

$$u_t = F(x, u, u_1, u_2, \cdots, u_m) = F[x, u], \tag{3}$$

where the u_i represent higher x-derivatives of u. Suppose this equation has a 1-parameter group of symmetries $u(x,t) \mapsto \tilde{u}(x,t;\tau)$, taking solutions to solutions, such that $\partial\tilde{u}/\partial\tau|_{\tau=0} = G(x, t, u, u_1, \cdots, u_n)$. Then the characteristic G must satisfy

$$D_t G = F_*(G), \tag{4}$$

where the linear differential operator F_* is the Fréchet derivative of F,

$$F_* = \sum_{k\geq 0} \frac{\partial F}{\partial u_k}(D_x)^k,$$

and D_t is the t-derivative computed using (3). Solving (4) to determine the symmetries of a given PDE is usually at least as hard as solving the original equation (3). However, since (3) should hold for all initial data $u(x)$, we get additional conditions on G by differentiating with respect to the u_k. This is equivalent to taking the Fréchet derivative of both sides of (4), and yields

$$D_t G_* = [F_*, G_*] + D_\tau F_*, \tag{5}$$

wherein the time derivatives are applied to the coefficients of F_*, G_* as polynomials in D_x. Thus, the terms in (5) are differential operators of orders n, $n+m-1$ and m respectively. A differential operator $\mathcal{R}$ of order n is called a generalized symmetry of rank k for (3) if the order of $D_t\mathcal{R} - [F_*, \mathcal{R}]$ is at most $n + m - k$. If its rank is arbitrarily high, then $D_t\mathcal{R} = [F_*, \mathcal{R}]$ and $\mathcal{R}$ is a recursion operator for (3), taking symmetry to symmetry. (In practice, one allows $\mathcal{R}$ to be a formal pseudo-differential operator, i.e., a finite polynomial in D_x plus a formal power series in D_x^{-1}.)

According to MSS [**7**], if (3) has a (formal) generalized symmetry of arbitrarily high rank, then it has a sequence of canonical conserved integrals $\int \rho_k \, dx$, $k \geq -1$, the first two of which have

$$\rho_{-1} = \left(\frac{\partial F}{\partial u_m}\right)^{-1/m}, \qquad \rho_0 = \frac{\partial F}{\partial u_{m-1}} \Big/ \frac{\partial F}{\partial u_m}.$$

Imposing the conservation laws associated to these densities means requiring that $D_t\rho_k$ be an exact x-derivative. We will see below that, in the case when (3) arises from an arclength-preserving third-order geometric evolution equation for planar curves, imposing ρ_0 is vacuous, while imposing ρ_{-1} guarantees (with one exception) at least formal integrability.

2. Classifying Flows for Planar Curves

For simplicity, we'll assume that the flow is arclength-preserving, so that it takes the form

$$\gamma_t = g\,T + f\,N, \qquad \text{where } f = \kappa^{-1} g', \tag{6}$$

where g is a function of κ and finitely many arclength derivatives. (To be consistent with the previous section, we'll re-label κ as $u(x,t)$, and its x-derivatives as u_1, u_2, etc.) When γ evolves by (6), its curvature satisfies

$$u_t = \mathcal{G}(g) \qquad \text{where } \mathcal{G} = D_x \circ u \circ \left((u^{-1}D_x)^2 + 1\right). \tag{7}$$

Let $\mathcal{F}_m$ denote the space of functions of u and its first m derivatives; then $\mathcal{G}(g) \in \mathcal{F}_m$ when $g \in \mathcal{F}_{m-3}$. Since only odd-order scalar evolution equations can have non-trivial conserved densities of arbitrarily high order[2], we will restrict attention to functions g which are even order in the derivatives of u. For evolution equations of the form (7), it is easy to calculate that

$$\rho_0 = D_x\left(3\ln\frac{\partial g}{\partial u_{m-3}} - 2\ln u\right) + \frac{\partial g}{\partial u_{m-4}} \Big/ \frac{\partial g}{\partial u_{m-3}}$$

when $g \in \mathcal{F}_{m-3}$ for $m > 3$, and that ρ_0 is an exact derivative when $g \in \mathcal{F}_0$.

2.1. Third-Order Flows. Here, we'll assume that g is a function of κ only; then, since the curvature itself is a second-order invariant of γ, then (6) gives a third-order flow for γ. Now (7) becomes a third-order scalar evolution equation,

$$u_t = (g(u)'/u)'' + (ug(u))'. \tag{8}$$

The first canonical density takes the form

$$\rho_{-1} = h(u)^{-1/3}, \qquad \text{where } h(u) = u^{-1}dg/du \neq 0.$$

Since $D_t\rho_{-1}$ must be an exact x-derivative, we obtain the necessary condition $\mathsf{E}(D_t\rho_{-1}) = 0$, where E is the Euler operator: $\mathsf{E} = \sum_{n\geq 0}(-D_x)^n \circ \partial/\partial u_n$. Subtracting off exact x-derivatives from $D_t\rho_{-1}$ yields a function $P(u, u_1, u_2)$, defined in terms of g and its u-derivatives, which must vanish for all initial data $u(x)$. Because this condition takes the form

$$9h(u)Q(u)u_1u_2 + \left[3\frac{d}{du}(h(u)Q(u)) - 10Q(u)\frac{dh}{du}\right]u_1^3 = 0,$$

where

$$Q(u) = 40\left(\frac{d}{du}h(u)\right)^3 - 45\,h(u)\left(\frac{d^2}{du^2}h(u)\right)\frac{d}{du}h(u) + 9\,(h(u))^2\,\frac{d^3}{du^3}h(u) = 0,$$

it follows that $Q(u)$ must be identically zero. Then $Q(u) = 0$ a third-order ODE for $h(u)$, with solution

$$h(u) = (au^2 + bu + c)^{-3/2}, \tag{9}$$

[2]See [**2**] for this result and its multivariable generalizations.

in turn defining $g(u)$ up to an arbitrary additive constant. (Since such constants add sliding along the curve to the evolution (6), they do not affect the shape of the evolving curve.)

It is rather surprising that, once conservation of ρ_{-1} is imposed, almost every flow (8) for $h(u)$ of the form (9) conserves the rest of the canonical densities for third-order equations given by MSS.[3] In some cases, the densities are more easily checked by first normalizing the equations by simple changes of variable, as we will now do. (Note, however, that while scaling in curvature u can be induced by scaling the underlying curve, even a simple change of variable like translation in u is not induced by a geometric transformation, and so will not preserve the form of (8).)

2.2. Normal Forms. In some instances, the evolution equations obtained above may be identified with well-known PDEs, through changes of variable and contact transformations. When we confine ourselves to real variables, the obvious invariants are the sign of $\Delta = b^2 - 4ac$, and whether or not a vanishes.

When $\Delta = 0$ but $a \neq 0$, we can assume $h(u) = (u - C)^{-3}$ by scaling time. Changing to the variable $v = u - C$, we obtain the normal form

$$v_t = D_x \left(v^{-3} v_2 - 3v^{-4} v_1^2 - \tfrac{3}{2} C v^{-1} - \tfrac{1}{2} C^2 v^{-2} \right).$$

Next, let $w = \int v dx$ be a potential, satisfying

$$w_t = w_1^{-3} w_3 - 3w_1^{-4} w_2^2 - \tfrac{3}{2} C w_1^{-1} - \tfrac{1}{2} C^2 w_1^{-2}.$$

Now perform a hodograph transformation, letting $\overline{w} = x$, $\overline{x} = w$ and $\overline{t} = t$. This gives

$$\overline{w}_t = \overline{w}_3 + \tfrac{3}{2} C (\overline{w}_1)^2 + \tfrac{1}{2} C^2 (\overline{w}_1)^3,$$

where subscripts now indicate partials with respect to $\overline{x}$. Letting $\overline{v} = \overline{w}_1$ gives

$$\overline{v}_t = \overline{v}_3 + 3C(\overline{v} + \tfrac{1}{2} C \overline{v}^2) \overline{v}_1.$$

When $C = 0$, this is linear; when $C \neq 0$, this can be transformed to the mKdV by completing the square in v', and then performing a Galilean boost.[4] If $\Delta = 0$ and $a = 0$, then we can assume $g(u) = \frac{1}{2} u^2$, which gives the mKdV equation immediately.

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[3]The density ρ_2 is printed incorrectly in [**7**], and should be

$$\rho_2 = -\tfrac{1}{3} F_2 D^2 \rho - F_1 D\rho + F_0 \rho + F_2 (D\rho)^2/\rho - \tfrac{1}{3} F_1 F_2 \rho^4 + \tfrac{1}{3} F_2^2 \rho^3 D\rho + \tfrac{2}{27} F_2^3 \rho^7 + \tfrac{1}{3} \rho \sigma_0,$$

where $\rho = \rho_{-1}$, $D = D_x$, $F_k = \partial F / \partial u_k$, and $D_t \rho_0 = D_x \sigma_0$.

[4]Although these transformations are part of the normalization procedure prescribed in [**7**], the fact that the $h(u) = u^{-3}$ equation could be linearized in this way was first brought to my attention by Danny Arrigo.

When $\Delta \neq 0$ and $a \neq 0$, we may scale to get $h(u) = v^{-3}$, where $v = \sqrt{k \pm (u-C)^2}$ for some constants C and $k \neq 0$. Then by following through transformations similar to the above, we can identify this equation with a sub-type of form (4.1.14) in [**7**]. If $a = 0$, scale to get $h(u) = v^{-3}$ for $v = \sqrt{u-C}$. Then

$$v_t = D_x \left(\frac{v_2}{v^3} - \frac{3v_1^2}{v^4} + 2\left(v + \frac{C}{v}\right)\right).$$

This equation is an exception, since it conserves canonical density ρ_1 only when $C = 0$; in that case, it can be identified as being of form (4.1.30) in [**7**]. Note that, when $C = 0$, $u_t = -2D_x^3(u^{-1/2}) + 2D_x(u^{3/2})$, which resembles one form of the Harry Dym equation, $u_t = D_x^3(u^{-1/2})$.

3. Third-Order Flows and Recursion Operators

Our goal in this section is to obtain bi-Hamiltonian structures for at least a representative sample of the third-order evolution equations obtained above. Such a structure comprises a pair of skew-adjoint (pseudo-)differential operators $\mathcal{D}$ and $\mathcal{E}$, such that

$$u_t = F[u] = \mathcal{D}\,\mathsf{E}\rho_0 = \mathcal{E}\,\mathsf{E}\rho_1, \tag{10}$$

(where ρ_0, ρ_1 are some non-trivial conserved densities for the flow—not necessarily the canonical densities of MSS) and such that $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{D} + \mathcal{E}$ each define Poisson structures. In our case, $\mathcal{D}$ will be of order 3 and $\mathcal{E}$ of order 1.

These operators are used formally to obtain higher-order conserved densities ρ_n such that

$$\mathsf{E}\rho_{n+1} = \mathcal{E}^{-1}\mathcal{D}\,\mathsf{E}\rho_n, \tag{11}$$

while a hierarchy of flows which commute with (10) can be defined by

$$u_t = K_n, \qquad K_{n+1} = \mathcal{D}\mathcal{E}^{-1}K_n, \qquad K_0 = F[u].$$

Because these operators will typically involve antidifferentiation, it must be proved that the ρ_n and K_n are local functions, i.e., functions of u and finitely many derivatives. In addition, we will want to know if the higher-order flows $u_t = K_n$ in the hierarchy are geometric.

3.1. Example 1.

The curve flow generated by $g(u) = -1/u$ is

$$\gamma_t = -\frac{1}{\kappa}T + \frac{\kappa_x}{\kappa^3}N, \tag{12}$$

and the corresponding evolution equation[5] for curvature is

$$u_t = D_x^2(u^{-3}u_1) = -\tfrac{1}{2}D_x^3(u^{-2}). \tag{13}$$

Although, as indicated above, this equation may be transformed to $u_t = u_{xxx}$, the bi-Hamiltonian structure for this equation is of independent interest, both because it may serve as a guide to discovering the structure for other members of the family generated by (9), and because the higher-order densities may be derived in a geometric way.

The *planar evolute* [**11**] of γ, defined by $\overline{\gamma} = \gamma + \kappa^{-1}N$, has arclength coordinate $\bar{x} = -\kappa^{-1}$ and curvature

$$\overline{\kappa} = \kappa^3/\kappa_x. \tag{14}$$

Moreover, up to reparametrization in x, the evolute evolves by the same flow:

$$\partial\overline{\gamma}/\partial t \cdot \overline{N} = (\overline{\kappa})^{-3}\partial\overline{\kappa}/\partial\overline{x}.$$

It follows that any density, expressed in terms of $\overline{\kappa}$, whose integral is conserved by flow of $\overline{\gamma}$, corresponds under the substitution (14) to a higher-order conserved density (for the same flow) along γ. For example,

$$\int (\overline{\kappa})^{-1} d\overline{x} = \int \kappa^{-5}(\kappa_x)^2 dx$$

gives the next conserved density after $\int \kappa^{-1}dx$ for this flow, and so on.

Proposition 3.1. Define the functions a_n by

$$a_0 = -1, \qquad a_{n+1} = D_x(a_n/\kappa).$$

Then $\rho_n = a_n^2/\kappa$ is a conserved density for (12)

Proof. Let $\overline{a}_n$ denote the equivalent functions computed on the evolute. Then it is easy to show by induction that $\bar{a}_n = -\kappa^2 a_{n+1}/\kappa_x$, and it follows that $\overline{\rho}_n d\overline{x} = \rho_{n+1}dx$. ☐

Because passage from the evolute $\overline{\gamma}$ to the involute γ transforms curves which are critical for $\int \rho_n dx$ to those critical for $\int \rho_{n+1}dx$, this transformation behaves like a Bäcklund transformation. In fact, it can be written as a Bäcklund transformation for the PDE (13), albeit one which includes a change in the space variable to match the change in arclength parameter.

Using the above proposition, we obtain a sequence of conserved densities:

$$\rho_0 = \frac{1}{u}, \qquad \rho_1 = \frac{u_1^2}{u^5}, \qquad \rho_2 = \frac{1}{u}\left(\frac{u_2}{u^3} - 3\frac{u_1}{u^4}\right)^2, \; \cdots$$

[5]This equation also arises as a rescaling limit of the Casimir flow in an integrable hierarchy, the compacton 'dual' to the mKdV hierarchy, constructed by Olver and Rosenau [**9**].

where we have reverted to writing u in place of curvature κ. Applying the Euler operator produces a sequence of characteristics for the conservation laws:

$$Q_0 = -\frac{1}{u^2},$$
$$Q_1 = -2\frac{u_2}{u^5} + 5\frac{u_1^2}{u^6},$$
$$Q_2 = 2\frac{u_4}{u^7} - 28\frac{u_1u_3}{u^8} - 21\frac{u_2}{u^4} + 196\frac{u_1^2u_2}{u^9} - 189\frac{u_1^4}{u^{10}}, \ \ldots$$

Applying the operator $\mathcal{D} = D_x^3$ then produces a sequence of functions K_n such that the flows $u_t = K_n[u]$ commute with (13).

To determine the recursion operator $\mathcal{E}^{-1} \circ \mathcal{D}$ for the Q_n, begin by 'noticing' that the quantities $L_n = D_x^2(Q_i)$ are related by $L_{n+1} = -\left(D_x \circ u^{-1}\right)^2 L_n$. It follows that the characteristics are related formally by

$$Q_{n+1} = -D_x^{-1}\left(u^{-1}D_x\right)^2 D_xQ_n. \tag{15}$$

LEMMA 3.2. The quantities Q_n defined by this relation, starting with $Q_0 = -1/u^2$, are all local functions of u and its derivatives.

PROOF. We need to prove that the antidifferentiation operator in (15) always produces a local function, i.e., that $(u^{-1}D_x)^2 D_xQ_n = u^{-1}D_x(u^{-1}L_n)$ is an exact derivative.

Let $w = u^{-1}$ and note that $L_0 = -2D_x(ww_x)$, and in general $L_n = -2D_x \circ (wD_x)^{2n+1}w$. Then, using integration by parts,

$$\begin{aligned} \mathsf{E}\left[wD_x(wL_n)\right] &= -2\mathsf{E}\left[w(D_x \circ w)^{2n+2}D_xw\right] \\ &= -2(-1)^{n+1}\mathsf{E}\left[\left((wD_x)^{n+1}w\right) D_x \left((wD_x)^{n+1}w\right)\right] = 0 \end{aligned}$$

□

Since $\mathcal{D} = D_x^3$ involves only differentiation, it follows that the functions $K_n = \mathcal{D}Q_n$ are all local. Comparing (15) with (11) shows that the other operator is given by

$$\mathcal{E} = -D_x \circ uD_x^{-1} \circ uD_x. \tag{16}$$

This is clearly skew-adjoint. Furthermore, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{D}+\mathcal{E}$ all define Poisson brackets which satisfy the Jacobi identity. (This can be verified using the calculus of functional multivectors [**8**]; the calculations are omitted for reasons of space.) It then follows that the flows $u_t = K_n[u]$ are mutually commuting and that the Q_i are characteristics of conservation laws for all these flows ([**8**], Thm. 7.24). Finally, the following proposition, which is easy proved by induction, shows that the flows $u_t = K_n = D_xL_n$ are all geometric.

Proposition 3.3. Let $g_0 = -2/u$ and $g_{n+1} = -(u^{-1}D_x)^2 g_n - 2/u$. Then the quantities L_n defined by the above recurrence relation, starting with $L_0 = D_x^2(-1/u^2)$, satisfy $L_n = 2 + u(1+(u^{-1}D_x)^2)g_n$, so that $K_n = \mathcal{G}(g_n)$.

3.2. Example 2. Here we consider the flow generated by $g(u) = -1/\sqrt{1+u^2}$. For this flow, we can find simplified characteristics by linearly combining the variational derivatives of the MSS densities:

$$\begin{aligned} Q_0 &= \mathsf{E}(\rho_{-1}) = u/v \\ Q_1 &= \tfrac{1}{3}\mathsf{E}(\rho_1 + \tfrac{1}{2}\rho_{-1}) = \frac{u_2}{v^5} - \frac{5}{2}\frac{uu_1^2}{v^7} - \frac{1}{2}\frac{u}{v^3} \\ Q_2 &= \tfrac{1}{9}\mathsf{E}(\rho_3 - \tfrac{1}{2}\rho_1) \qquad \text{where } v = \sqrt{1+u^2}. \end{aligned}$$

Note that since the PDE here is given by

$$u_t = (D_x^3 - D_x)Q_0,$$

we will chose the skew-adjoint operator $\mathcal{D} = D_x^3 - D_x$ in this case. Next, one observes that the above characteristics satisfy

$$Q_{n+1} = \frac{u}{v} D_x^{-1} \circ v^{-1} D_x \circ u^{-1}(D_x^2 - 1)Q_n,$$

leading, by comparison with (11), to

$$\mathcal{E} = D_x \circ u D_x^{-1} \circ v D_x \circ \frac{v}{u}.$$

(Note that this would agree, up to sign, with (16) if v were replaced by u.) Again, we can verify that the Jacobi identities hold for $\mathcal{D}, \mathcal{E}$ and $\mathcal{D}+\mathcal{E}$; then, one can repeatedly apply the recursion operator $\mathcal{R} = \mathcal{D}^{-1} \circ \mathcal{E}$ to formally generate commuting flows $u_t = K_n[u]$. However, as of this writing, it is not known if these are always local functions, or if the flows are always geometric. (The former question might be approached by proving that $\mathcal{R}$ is hereditary, then appealing to the results of Sanders and Wang [**10**].)

4. Further Research

4.1. Higher-Order Flows. Are there any more H-integrable geometric evolution equations for planar curves, other than the ones already discussed, along with their higher-order commuting flows? A first step in answering this would be to classify those equations of the form

$$u_t = \mathcal{G}(g), \qquad g = g(u, u_1, u_2), \tag{17}$$

which give a fifth-order evolution equation for the curvature u. Although MSS [**7**] only classified equations of the form $u_t = u_5 + f(x, u, \cdots, u_4)$—by dint of calculating the canonical densities ρ_k, $k \geq 0$, and imposing the condition that they be conserved—it is not difficult to follow their methods and generate canonical densities for general fifth-order equations.

Investigations of flows of the form (17) are ongoing. However, the following results have been obtained so far:

(1) If canonical densities ρ_{-1} and ρ_0 are conserved, then g is linear in u_2.
(2) There is a family of analytic generators g, depending locally on two functions of one variable, for which $\partial^2 g/\partial u_1 \partial u_2 \neq 0$ and for which (17) conserves ρ_{-1} and ρ_0.
(3) If densities ρ_{-1}, ρ_0 and ρ_1 are conserved, then $\partial^2 g/\partial u_1 \partial u_2 = 0$ and g belongs to one of two distinct finite-dimensional families of generators.

(Details of this classification will appear elsewhere.) One of the aforementioned families contains the fifth-order flows commuting with the third-order flows of section 2.1, distinguished by the presence of the square root of a general quadratic. The other family contains cube roots; one example is generated by

$$g = u^{-1/3}\left(\frac{u_2}{u^2} - \frac{4}{3}\frac{u_1^2}{u^3} + Cu\right), \qquad C = 6 \text{ or } 3/8.$$

It is not known yet, however, if any of the latter equations are H-integrable.

4.2. Companion Flows in $\mathbb{R}^3$. One of the interesting features of the hierarchy of geometric evolution equations which commute with the vortex filament flow (1) is that every second flow in the hierarchy preserves planarity [**4**]. These flows include

$$\gamma_t = \tfrac{1}{2}\kappa^2 T + \kappa' N + \kappa\tau B, \tag{18}$$

which gives the planar filament flow (2) when $\tau = 0$. It is natural to ask if there is an analogous integrable flow for curves in $\mathbb{R}^3$ which restricts to give one of the flows discussed in §3.

In the case of (12), the geometry provides a hint. The non-planar evolutes [**11**] of a planar curve γ are defined by

$$\overline{\gamma} = \gamma + \frac{1}{\kappa}(N + \alpha B), \qquad \alpha \in \mathbb{R}.$$

When γ evolves by (12), the evolutes flow by

$$\gamma_t = -\frac{1}{\kappa}T + \frac{\kappa'}{\kappa^3}N + \frac{\tau}{\kappa^2}B, \tag{19}$$

up to rescaling in time by a factor of $(1+\alpha^2)^{3/2}$. For the rest of this section we will study the integrability of (19).

In general, when a curve satisfies $\gamma_t = W$, its curvature and torsion evolve by

$$\begin{aligned}\kappa_t &= N\cdot W'' - 2\kappa T\cdot W' \\ \tau_t &= \left(\kappa^{-1}B\cdot W''\right)' + (\kappa B - \tau T)\cdot W'\end{aligned} \tag{20}$$

(see [**3**] for a derivation). Substituting the right-hand side of (19) for W gives

$$\kappa_t = D_x\left((\kappa'/\kappa^3)' - \tfrac{3}{2}\tau^2/\kappa^2\right) \tag{21a}$$

$$\tau_t = D_x\left((\tau'/\kappa^3)' - \tau^3/\kappa^3\right) \tag{21b}$$

One may calculate the conserved densities for this system that are at most first order in the derivatives of κ and τ. These fall into two groups, of scaling weight one

$$\lambda_0 = \tau, \quad \lambda_1 = \kappa, \quad \lambda_2 = -\frac{\tau^2}{2\kappa}, \quad \lambda_3 = \frac{\tau^2(\kappa')^2 + \kappa^2(\tau')^2 - 2\kappa\tau\kappa'\tau}{2\kappa^5} - \frac{\tau^4}{8\kappa^3}$$

and scaling weight zero

$$\mu_1 = -\frac{\tau}{\kappa}, \quad \mu_2 = \frac{\tau(\kappa')^2}{\kappa^5} - \frac{\tau'\kappa'}{\kappa^4} - \frac{\tau^3}{2\kappa^3}.$$

Moreover, there are two sequences of geometric flows which commute with (19), and which can be obtained by applying either of two Hamiltonian operators to the conserved densities.

First, note that in (19) the binormal and tangential components coincide with those of $\mathsf{E}\mu_1$, where now the Euler operator on differential functions of $u=\kappa$ and $v=\tau$ produces a two-component vector:

$$\mathsf{E} = \left[\sum_{n\ge 0}(-D_x)^n\circ\partial/\partial u_n,\ \sum_{n\ge 0}(-D_x)^n\circ\partial/\partial v_n\right].$$

More generally, we may define a map from densities to arclength-preserving geometric flows by

$$\mathcal{X}(\rho) = \mathcal{P}\left(\mathsf{E}(\rho)\cdot[B,T]\right),$$

where the operator $\mathcal{P}$ is defined by $\mathcal{P}(a\,T + b\,B) = a\,T + \kappa^{-1}a'N + b\,B$. Then $\mathcal{X}(\mu_1)$ gives the right-hand side of (19), while applying $\mathcal{X}$ to other densities produces the following vector fields:

$$\mathcal{X}(\tau) = T, \quad \mathcal{X}(\kappa) = B, \quad \mathcal{X}(\lambda_1) = -\frac{\tau}{\kappa}T - \frac{1}{\kappa}\left(\frac{\tau}{\kappa}\right)'N + \frac{\tau^2}{2\kappa^2}B.$$

In general, let $X_k = \mathcal{X}(\lambda_k)$ and $Y_k = \mathcal{X}(\mu_k)$. It can be verified that T, B, X_1, X_2, X_3, Y_1 and Y_2 induce mutually commuting systems of evolution equations for κ and τ.

Such equations are derived, in general, by setting $W = \mathcal{X}(\rho)$ in (20). This produces $(\partial_t\kappa, \partial_t\tau) = \mathcal{D}\,\mathsf{E}\rho$, where $\mathcal{D}$ is the skew-adjoint third-order differential operator defined by

$$\mathcal{D} = \begin{pmatrix} \mathcal{A}^* - \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & \mathcal{C} - \mathcal{C}^* \end{pmatrix}$$

with $\mathcal{A} = \tau \circ D_x$, $\mathcal{B} = (D_x^2 - \tau^2) \circ \kappa^{-1} D_x + D_x \circ \kappa$, and $\mathcal{C} = \mathcal{A} + D_x \circ (\tau/\kappa) D_x \circ \kappa^{-1} D_x$.

As in §3, we expect these flows to be Hamiltonian with respect to a first-order operator $\mathcal{E}$. In particular, the third-order flow $\mathcal{D}\,\mathsf{E}\mu_1$ should be produced by applying $\mathcal{E}$ to some combination of $\mathsf{E}\lambda_3$ or $\mathsf{E}\mu_2$, both of which are second-order. In fact, comparing the second component of the latter,

$$\mathsf{E}_\tau \mu_2 = \frac{\kappa''}{\kappa^4} - 3\frac{(\kappa')^2}{\kappa^5} - \frac{3\tau^2}{2\kappa^3},$$

with (21a) shows that $\kappa_t = [0, D_x \circ \kappa] \cdot \mathsf{E}\mu_2$. In order to complete this to a skew-adjoint matrix differential operator applied to $\mathsf{E}\mu_2$, we must write

$$\tau_t - \kappa D_x(\mathsf{E}_\kappa \mu_2) = 2\left[\frac{\kappa'''}{\kappa^4} - 10\frac{\kappa''\kappa'}{\kappa^5} + 15\frac{(\kappa')^3}{\kappa^6} + \frac{9\tau^2\kappa'}{2\kappa^4}\right]\tau + \left[\frac{\kappa''}{\kappa^4} - 3\frac{(\kappa')^2}{\kappa^5} - \frac{15\tau^2}{2\kappa^3}\right]\tau'$$

as a skew-adjoint scalar first-order differential operator applied to $\mathsf{E}_\tau\mu_2$. It is not hard to see that the correct operator is $\tau D_x + D_x \circ \tau$, giving

$$\mathcal{E} = \begin{pmatrix} 0 & D_x \circ \kappa \\ \kappa D_x & \tau D_x + D_x \circ \tau \end{pmatrix}.$$

Then $\mathcal{D}\mathsf{E}\lambda_1 = \mathcal{E}\mathsf{E}\lambda_2$, $\mathcal{D}\mathsf{E}\lambda_2 = \mathcal{E}\mathsf{E}(\lambda_3 + \lambda_2)$, while $\mathcal{E}\mathsf{E}\tau = \mathcal{D}\mathsf{E}\tau$ and $\mathcal{E}\mathsf{E}\lambda_1 = \mathcal{E}\mathsf{E}\mu_1 = 0$.

The operators $\mathcal{D}$ and $\mathcal{E}$ have been discovered independently by Mari Beffa, Sanders and Wang [**6**], who are able to go much further in proving the integrability of (19) and its generalizations to constant-curvature space forms. For example, in order to see that $\mathcal{D}$ and $\mathcal{E}$ give the system (21) a bi-Hamiltonian structure, it must be verified that $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{D} + \mathcal{E}$ satisfy the Jacobi identity. These authors, in fact, embed $\mathcal{D}$ and $\mathcal{E}$ in a two-parameter linear span of Hamiltonian operators, thus endowing the flow with a tri-Hamiltonian structure. Furthermore, they assert that $\mathcal{R} = \mathcal{D} \circ \mathcal{E}^{-1}$ is a hereditary operator, implying that it continues to produce systems of evolution equations which commute with (21). Furthermore, the above derivation of operator $\mathcal{D}$ shows that all of these are induced by geometric flows for curves in $\mathbb{R}^3$.

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