LOCAL EXISTENCE OF RICCI SOLITONS

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Introduction

The Ricci flow $\partial g/\partial t = -2\,\mathrm{Ric}(g)$ is an evolution equation for Riemannian metrics. It was introduced by Richard Hamilton, who has shown in several cases ([7], [8], [9]) that the flow converges, up to re-scaling, to a metric of constant curvature. However, "soliton" solutions to the flow give examples where the Ricci flow does not uniformize the metric, but only changes it by diffeomorphisms. Soliton solutions are generated by initial data satisfying

$$\mathcal{L}_X g = -2\,\mathrm{Ric}(g)$$

for some complete vector field X. The subsequent solution of the flow is $g_t = \phi_t^*(g)$, where ϕ_t is the 1-parameter family of diffeomorphisms generated by X.

If we wish to classify Ricci solitons, it is helpful to start with the following questions:

> How large is the space of Ricci soliton metrics, modulo diffeomorphisms? For example, do they lie in a finite-dimensional space?
>
> Are there any integrability conditions implied by the soliton condition that may help us classify solitons?

In this paper we will use the techniques of exterior differential systems to answer these questions. In particular, it will turn out that the soliton condition is involutive, and, modulo diffeomorphisms, the n-dimensional soliton metrics depend locally on n^2+n arbitrary functions of $n-1$ variables. We will also investigate the differential relations between a soliton metric g and the corresponding vector field that are implied by the Ricci soliton condition.

We will begin with a brief explanation of involutivity.

The meaning of involutivity

A system of k-th order PDE

$$F^\rho(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^i}, \dots, \frac{\partial^k u^\alpha}{\partial x^I}) = 0, \quad |I| = k, \quad \rho = 1\dots r$$

for s unknown functions u^α of $x^1, \dots, x^n$ determines, and is in fact equivalent to, a submanifold $\mathcal{R}$ in the space $J^k(\mathbb{R}^n, \mathbb{R}^s)$ of k-jets of functions from $\mathbb{R}^n$ to $\mathbb{R}^s$. In fact if x^i, u^α, p^α_i, $p^\alpha_{ij}, \dots, p^\alpha_I$ are the usual coordinates on $J^k(\mathbb{R}^n, \mathbb{R}^s)$ then $\mathcal{R}$ is cut out by the functions

$$F^\rho(x^i, u^\alpha, p^\alpha_i, \dots, p^\alpha_I) = 0, \quad \rho = 1\dots r$$

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A solution of the PDE corresponds to an integral manifold $N \subset \mathcal{R}$ of the contact system on $J^k(\mathbb{R}^n, \mathbb{R}^s)$. We can also define the *prolongation* $\mathcal{R}^{(k+1)} \subset J^{k+1}(\mathbb{R}^n, \mathbb{R}^s)$ of $\mathcal{R} = \mathcal{R}^{(k)}$ as being cut out by the functions F^ρ and their total derivatives $\dfrac{DF^\rho}{Dx^i}$, $i = 1 \dots n$; similarly we can define $\mathcal{R}^{(k+2)}$, $\mathcal{R}^{(k+3)}$, etc.

The "forgetful functors" $\pi_l : J^{l+1}(\mathbb{R}^n, \mathbb{R}^s) \to J^l(\mathbb{R}^n, \mathbb{R}^s)$, which remember only the l-jet, are submersions, and of course $\pi_l(\mathcal{R}^{(l+1)}) \subset \mathcal{R}^{(l)}$. However, there is no guarantee that $\mathcal{R}^{(l+1)}$ submerses onto $\mathcal{R}^{(l)}$ via π_l. Put another way, there is no guarantee that an l-jet solution can be extended to an $(l+1)$-jet solution of the PDE.

To give a simple example, consider the following PDE for functions u and v of x and y:

$$\begin{aligned} u_x &= v + f_1(x,y) \\ u_y &= f_2(x,y) \end{aligned}$$

These define a smooth manifold $\mathcal{R}^{(1)} \subset J^1(\mathbb{R}^2, \mathbb{R}^2)$ of codimension two. $\mathcal{R}^{(2)}$ is cut out by these equations and their derivatives

$$\begin{aligned} u_{xx} &= v_x + \partial_x f_1 \\ u_{xy} &= v_y + \partial_y f_1 \\ u_{xy} &= \partial_x f_2 \\ u_{yy} &= \partial_y f_2 \end{aligned}$$

which imply that $v_y + \partial_y f_1 = \partial_x f_2$, i.e. $\mathcal{R}^{(2)}$ lies over a submanifold of codimension *three* in $J^1(\mathbb{R}^2, \mathbb{R}^2)$.

When a system of PDE becomes involutive at the k-jet level, each $\mathcal{R}^{(l+1)}$ submerses onto $\mathcal{R}^{(l)}$ for $l \geq k$, and the rank of the submersion can be calculated (see [2], Thm IV.4.4). In fact, involution implies that not only is there no obstruction to extending a kth degree Taylor polynomial to a formal power series solution, but (in the case when the functions F^ρ are analytic) the power series will converge on a small neighbourhood in $\mathbb{R}^n$.

Ellipticity of the soliton condition

We will consider a slightly generalized soliton condition

$$\mathcal{L}_X \, g = 2 \, \mathrm{Rc}(g) + \lambda \, g \tag{1}$$

where λ is some constant. In this section, we will show this is an elliptic equation.

As it stands the condition is underdetermined; and, since it is invariant under diffeomorphisms, the solution space is at least as big as n arbitrary functions of n variables. To get a determined equation, and mod out by the diffeomorphism group, we adjoin the condition that the local coordinates x^i be harmonic functions with respect to the metric[1]. This is actually a trace condition on the Christoffel symbols: $g^{ij}\Gamma^k_{ij} = 0$. Since any new harmonic coordinate is a solution of an elliptic equation, the space of diffeomorphisms that preserve this condition is finite-dimensional. Putting this together with (1) we get a system that is, by a naive

[1] This fairly well-known trick is due to DeTurck; see [1], Chapter 5.

count, determined, first order in X, and of mixed second-order and first-order in g:

$$\begin{cases} 2R_{ij}(g) + \lambda\, g_{ij} - \nabla_i X_j - \nabla_j X_i = 0 \\ g^{ij}\Gamma^k_{ij} = 0 \end{cases} \tag{2}$$

(Here X_i denotes components of the metric dual of X.) Although the principal symbol of this system is degenerate, it turns out to be elliptic in the generalized sense of Douglis-Nirenberg [4]. This implies that g and X will be analytic in these coordinates. Since any other harmonic coordinates y^i satisfy an elliptic equation, they will also be analytic functions of the coordinates. We conclude

Theorem 1. If M has a metric g satisfying (1), then M has the structure of a real analytic manifold with respect to which g and X are analytic.

Proof. To check the Douglis-Nirenberg condition, we choose the parts of the first set of equations in (2) that are second-order in g and first-order in X, and the first-order part of the second set of equations. The kernel of the resulting symbol mapping is given by

$$s_{ij}|\xi|^2 + g^{kl}s_{kl}\xi_i\xi_j - \xi^l(s_{il}\xi_j + s_{jl}\xi_i) + \xi_i t_j + \xi_j t_i = 0 \tag{3}$$

$$2s_{ij}\xi^j - g^{kl}s_{kl}\xi_i = 0. \tag{4}$$

(Here, s_{ij} and t_i are components of a symmetric n by n matrix and an n-vector in the kernel, $\xi^l = g^{kl}\xi_k$, and $|\xi|^2 = g^{kl}\xi_k\xi_l$.) We want to show that this kernel is zero for all real covectors $\xi \neq 0$.

Contracting (3) with ξ^j shows that $t_i = a\xi_i$ for a scalar a, while contracting (3) with g^{ij} gives $g^{kl}s_{kl}|\xi|^2 - s_{kl}\xi^k\xi^l + a|\xi|^2 = 0$. Contracting (4) with ξ^i shows that $g^{kl}s_{kl}|\xi|^2 = 2s_{kl}\xi^k\xi^l$, and hence $a = -\frac{1}{2}g^{kl}s_{kl}$. Now (3) reads

$$s_{ij}|\xi|^2 - \xi^l(s_{il}\xi_j + s_{jl}\xi_i) = 0. \tag{5}$$

Substituting in from (4) gives $s_{ij}|\xi|^2 = g^{kl}s_{kl}\xi_i\xi_j$. Now substituting $s_{ij} = b\xi_j\xi_j$ in (5) shows that $b = 0$. □

We now proceed to set up the exterior differential system that will allow us to further investigate the soliton condition.

The soliton system and its characteristic variety

Let $x^1 \dots x^n$ be local coordinates on an open set $U \subset \mathbb{R}^n$, and let ω^i be shorthand for $d(x^i)$. Let $\Omega = \omega^1 \wedge \dots \wedge \omega^n$, and let $\omega_{(i_i \dots i_k)}$ be the wedge product of the ω's such that

$$\omega_{(i_i \dots i_k)} \wedge \omega^{i_1} \wedge \dots \wedge \omega^{i_k} = \Omega;$$

for example, $\omega_{(2)} = \omega^3 \wedge \omega^1$ when $n = 3$ and $\omega_{(41)} = -\omega^2 \wedge \omega^3$ when $n = 4$. It is easy to verify that

$$\begin{aligned} \omega_{(j)} \wedge \omega^i &= \delta^i_j \Omega \\ \omega_{(jk)} \wedge \omega^i &= \delta^i_j \omega_{(k)} - \delta^i_k \omega_{(j)} \\ \omega_{(ijk)} \wedge \omega^p &= \delta^p_i \omega_{(jk)} - \delta^p_j \omega_{(ik)} + \delta^p_k \omega_{(ij)} \end{aligned}$$

We will now set up our exterior differential system. Let U be an open set in $\mathbb{R}^n$, let S_+ denote the open set of positive definite symmetric $n \times n$ matrices g_{ij} and let C be a $n\binom{n+1}{2}$-dimensional vector space with coordinates $\Gamma^i_{jk} = \Gamma^i_{kj}$. Let g^{ij}

denote the inverse of g_{ij}, and let $R \subset C$ be the codimension-n subspace cut out by $g^{ij}\Gamma^k_{ij} = 0$. On $U \times S_+ \times R$ define the forms

$$\begin{aligned} \phi^i_j &= \Gamma^i_{jk}\omega^k \\ \gamma^{ij} &= dg^{ij} + g^{ik}\phi^j_k + g^{kj}\phi^i_k \\ \Phi^i_j &= d\phi^i_j + \phi^i_k \wedge \phi^k_j \\ \Phi^{ij} &= \Phi^i_k g^{kj} \end{aligned}$$

A metric on U can be thought of as a section of $U \times S_+$, equivalently a n-dimensional submanifold which submerses onto U, equivalently one on which the n-form Ω does not vanish. If we have a section of $U \times S_+ \times C$ on which the forms γ^{ij} vanish, then the Γ^i_{jk} are the Christoffel symbols of the metric, $\Phi^i_j = \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l$ are the curvature forms, and one computes that the the Ricci tensor appears in the $(n-1)$-form

$$\Phi^{ij} \wedge \omega_{(pij)} = R^j_j \omega_{(p)} - 2R^j_p \omega_{(j)}.$$

Let f_i and f_{ij} be coordinates on $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$; we define the following forms on $N = U \times S_+ \times R \times \mathbb{R}^n \times \mathbb{R}^{n^2}$:

$$\begin{aligned} \sigma_i &= df_i - f_j\phi^j_i - f_{ij}\omega^j \\ \Theta_p &= \Phi^{ij} \wedge \omega_{(pij)} + 2(f_{pj} + f_{jp})g^{jk}\omega_{(k)} - 2g^{ij}f_{ij}\omega_{(p)} \end{aligned}$$

If these forms vanish on a section of N, then the f_i are components of a 1-form whose metric dual X satisfies the soliton condition (1). (In what follows we will take $\lambda = 0$ for simplicity; however, all our conclusions carry through without change when $\lambda \neq 0$.) For, if these forms vanish, then $f_{ij} = \nabla_j X_i$ and

$$Rg_{ij} - 2R_{ij} + 2(f_{ij} + f_{ji}) - 2g^{kl}f_{kl}g_{ij} = 0.$$

We can trace this equation to eliminate the scalar curvature R, and then we get (1) in component form.

To apply the techniques of exterior differential systems, we will need to construct the differential ideal containing these forms; this ideal will be generated algebraically by the forms $\gamma^{ij}, \sigma_i, \Theta_i$ and their exterior derivatives. To simplify computations, we define

$$\begin{aligned} \tau_{ij} &= df_{ij} - f_{ik}\phi^k_j - f_{kj}\phi^k_i \\ \Psi^{ij} &= \Phi^{ij} + \Phi^{ji}. \end{aligned}$$

Then

$$\begin{aligned} d\gamma^{ij} &\equiv \Psi^{ij} \quad \mod\{\gamma^{ij}\} \\ d\sigma_i &\equiv -\tau_{ij} \wedge w^j - f_p\Phi^p_i \quad \mod\{\gamma^{ij}, \sigma_i\} \\ d\Theta_i &\equiv 2g^{jk}(\tau_{ij}w_{(k)} + \tau_{ji}w_{(k)} - \tau_{jk}\omega_{(i)}) \quad \mod\{\gamma^{ij}, \sigma_i, \Psi^{ij}\}. \end{aligned}$$

(In fact, if the vanishing of Θ_i corresponds to prescribing the Ricci tensor, then the vanishing of $d\Theta_i$ corresponds to the second Bianchi identity $g^{jk}(\nabla_i R_{jk} - 2\nabla_k R_{ij}) = 0$.)

By definition, an integral n-plane at a point $x \in N$ is an n-dimensional subspace of T_xN to which the forms in our system,

$$\gamma^{ij}, \sigma_i, \Psi^{ij}, d\sigma_i, \Theta_i, d\Theta_i, \tag{6}$$

restrict to be zero. We will only consider those integral n-planes E which satisfy the independence condition $\Omega|_E \neq 0$. Even so, at every point of N there are integral n-planes. This is because such an integral element is in fact a 2-jet of a metric and a 2-jet of a vector field satisfying (2). We can always find a metric on a small neighbourhood with prescribed curvature at one point by using power series. We can then obtain harmonic coordinates with the same 1-jet as our given coordinates in a neighbourhood of that point.

Given an integral n-plane $E \subset T_xN$ we can obtain a coframe ω^i, γ^{ij}, π^i_{jk}, σ_i, σ_{ij} for the larger space $U \times S_+ \times C \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ at x, with $\pi^i_{jk} = \pi^i_{kj}$, $\pi^i_{jk} \equiv d\Gamma^i_{jk} \mod \omega^i$, $\sigma_{ij} \equiv \tau_{ij} \mod \omega^i$, such that

$$\begin{cases} g^{jk}\pi^i_{jk}|_{T_xN} = 0 \\ \Psi^{ij} = (g^{jk}\pi^i_{kl} + g^{ik}\pi^j_{il}) \wedge \omega^l \\ d\sigma_i \equiv \sigma_{ij} \wedge \omega^j + f_p \pi^p_{il} \wedge \omega^l \\ \Theta_i = g^{kp}\pi^j_{pl} \wedge \omega^l \wedge \omega_{(ijk)} \\ d\Theta_i \equiv 2g^{jk}(\sigma_{ij} w_{(k)} + \sigma_{ji} w_{(k)} - \sigma_{jk}\omega_{(i)}) \end{cases} \tag{7}$$

and π^i_{jk}, σ_{ij} vanish on E. We will refer to this as a torsion-absorbed coframe. On any other integral n-plane at x the 1-forms of (6) vanish and $\pi^i_{jk} = s^i_{jkl}\omega^l$, $\sigma_{ij} = t_{ijk}\omega^k$, where the s^i_{jkl} and t_{ijk} must satisfy some homogeneous linear equations dictated by (7). Thus, the space of integral n-planes naturally forms an affine bundle over N; we will calculate the rank of this bundle later.

A hyperplane in E is characteristic if it is contained in more than one integral n-plane. The characteristic variety Ξ_E is the set of directions in $\mathbb{P}(E^*)$ that cut out characteristic hyperplanes, and the complex characteristic variety $\Xi^{\mathbb{C}}_E$ is cut out by the same equations considered over $\mathbb{C}$. There is a simple-minded algorithm for calculating the characteristic variety by duality.

Proposition 2. Let E be an integral n-plane of an arbitrary exterior differential system and let ω^i, π^a be a coframe with $\pi^a|_E = 0$. Given a nonzero $\xi \in E^*$ (expressed as a linear combination of the $\omega's$), for each form ψ in the system of degree at most n there is an expansion

$$\psi \wedge \xi = \sum_I \pi^\psi_I \wedge \omega^I + \dots$$

where for each multi-index I, π^ψ_I is some linear combination of the π^a and we leave off wedge products of two or more π^a's. (If the system is linear there will be none of these anyway.) Then ξ is in the (deprojectivized) characteristic variety exactly when the span of all the 1-forms π^ψ_I, taken over all algebraic generators ψ of the system of degree at most n, is a proper subspace of the span of the π^a.

Proof. The hyperplane $\xi^\perp$ is characteristic when its polar space

$$H(\xi^\perp) = \{v \in T_xN | (v \lrcorner \psi)|_{\xi^\perp} = 0, \forall \text{ generators } \psi\}$$

is larger than E (cf. Defn. III.1.5 and V.1.1 in [2]). The condition on v is equivalent to

$$((v \lrcorner \psi) \wedge \xi)|_E = 0;$$

since ψ vanishes on E this is equivalent to

$$(v\lrcorner(\psi\wedge\xi))|_E = 0.$$

Using the above expansion for $\psi\wedge\xi$ yields

$$H(\xi^\perp) = \{v\in T_xN | v\lrcorner\pi^\psi_I = 0, \forall\psi, I\}.$$

□

Proposition 3. If $\xi\in E^*\otimes\mathbb{C}$ is characteristic for E an integral n-plane of (6), then $g^{ij}\xi_i\xi_j = 0$. Since g_{ij} is positive definite this implies there are no real characteristics.

Proof. Let $\xi^i = g^{ij}\xi_k$, $|\xi|^2 = g^{km}\xi_k\xi_m$, and

$$\rho_{ijk} = g_{ip}\pi^p_{jk} + g_{jp}\pi^p_{ik}.$$

(Note that $g^{jk}\pi^i_{jk} = 0$ implies $g^{jk}\rho_{jki} = 2g^{jk}\rho_{ijk}$.) The forms that cut out $H(\xi^\perp)$ comprise γ^{ij}, σ_i, and the 1-forms

$$\begin{aligned}
\psi_{ijlm} &= \xi_l\rho_{ijm} - \xi_m\rho_{ijl}\\
\sigma_{ijk} &= \xi_k(\sigma_{ij} + f_l\pi^l_{ij}) - \xi_j(\sigma_{ik} + f_l\pi^l_{ik})\\
\theta_{ij} &= \xi^k(\rho_{kij} - \rho_{ijk} + \pi^p_{kp}g_{ij}) - \xi_i\pi^k_{jk}\\
\alpha_i &= (\sigma_{ij} + \sigma_{ji})\xi^j - g^{jk}\sigma_{jk}\xi_i
\end{aligned}$$

which arise from wedging ξ with Ψ^{ij}, $d\sigma_i$, Θ_i, and $d\Theta_i$ respectively. One can calculate that if

$$\beta_{ij} = \frac{1}{2}(\theta_{ij} + \theta_{ji} + g^{km}(\psi_{ikjm} + \psi_{jkim})) = -\rho_{ijk}\xi^k + g_{ij}g^{lm}\rho_{lmk}\xi^k$$

then

$$\psi_{ijlm}\xi^m + \xi_l(\beta_{ij} - \frac{1}{n-2}g_{ij}g^{kl}\beta_{kl}) = -|\xi|^2\rho_{ijl};$$

also,

$$\xi^k(\sigma_{ijk} - \sigma_{jik} - \sigma_{kij}) + \xi_j\alpha_i - \xi_i\alpha_j \equiv |\xi|^2(\sigma_{ij} - \sigma_{ji}) \mod\{\pi^i_{jk}\}$$

and

$$\xi^k(\sigma_{ijk} + \sigma_{jik}) + \frac{1}{2}(\xi_i\alpha_j + \xi_j\alpha_i) \equiv |\xi|^2(\sigma_{ij} + \sigma_{ji}) - g^{lm}\sigma_{lm}\xi_i\xi_j \mod\{\pi^i_{jk}, \sigma_{ij} - \sigma_{ji}\}.$$

This shows that if $|\xi|^2 = g^{ij}\xi_i\xi_j \neq 0$, the span of the 1-forms $\psi_{ijlm}, \sigma_{ijk}, \theta_{ij}, \alpha_i$ is the same as that of π^i_{jk}, σ_{ij}, and $H(\xi^\perp) = E$. □

Testing for Involutivity

First we will calculate the rank of the bundle of integral n-planes over each point of N.

Proposition 4. At each point of N the space of integral n-planes of (6) has dimension

$$\binom{n+1}{2}^2 - n^2 - \binom{n+1}{2} + n\binom{n+1}{2} - n.$$

Proof. To obtain an integral n-plane, set $\pi^i_{jk} = g^{ip} s_{pjkl}\omega^l$, $\sigma_{ij} = t_{ijk}\omega^k$, where the t_{ijk} have no symmetries but $s_{ijkl} = s_{ikjl}$, which we abbreviate by saying $s_{ijkl} \in V \otimes S^2V \otimes V$. When we substitute into (7), the resulting linear equations for the s_{ijkl} and t_{ijk} are

$$g^{jk} s_{ijkl} = 0 \tag{8}$$

$$s_{ijkl} + s_{jikl} - s_{ijlk} - s_{jilk} = 0 \tag{9}$$

$$t_{ikl} - t_{ilk} = -f_p g^{pj}(s_{jikl} - s_{jilk}) \tag{10}$$

$$g^{jk}(s^l_{jli} - s^l_{jil}) - g^{jl}(s^k_{jli} - s^k_{jil}) + g^{lm}(s^j_{lmj} - s^j_{ljm})\delta^k_i = 0 \tag{11}$$

$$g^{jk}(t_{ijk} + t_{jik} - t_{jki}) = 0 \tag{12}$$

where $s^i_{jkl} = g^{ip}s_{pjkl}$. The linear map in (9) can be factored as

$$V \otimes S^2V \otimes V \xrightarrow{\cong} S^2V \otimes V \otimes V \overset{\wedge_{34}}{\twoheadrightarrow} S^2V \otimes \Lambda^2 V$$

where the first map is symmetrization on factors 1 and 2. The kernel of the second map is clearly $S^2V \otimes S^2V$; so, to solve (9) set

$$s_{ijkl} = b_{ijkl} + b_{ikjl} - b_{jkil}$$

for $b_{ijkl} \in S^2V \otimes S^2V$. Then (8) and (11) become

$$(2b_{ijkl} - b_{jkil})g^{jk} = 0 \quad \forall i, l \tag{13}$$

$$g^{jl}b_{ikjl} - g_{ik}g^{jl}g^{mp}b_{jmlp} = 0 \quad \forall i, k \tag{14}$$

For any $a_{ij} \in S^2_oV$ (i.e. such that $g^{ij}a_{ij} = 0$), $b_{ijkl} = (n-2)a_{ij}g_{kl} + 2g_{ij}a_{kl}$ is a solution of (13). Substituting this in the left-hand side of (14) yields $n(n-2)a_{ij}$. Meanwhile $b_{ijkl} = 2ng_{ij}g_{kl} + (n-2)(g_{il}g_{jk} + g_{jl}g_{ik})$ also satisfies (13) and substituting this into (14) yields $(n^2+n-2)(2-n)g_{ik}$. Together, these two calculations show that the kernel of the linear map in (13) surjects onto S^2V under the linear map in (14). Thus the dimension of the space of s_{ijkl} satisfying (8), (9), (11) is $\binom{n+1}{2}^2 - n^2 - \binom{n+1}{2}$.

On the other hand the linear map on the t_{ijk} in (10) is clearly surjective with kernel $V \otimes S^2V$ and it is easy to show that this kernel surjects onto V under the linear map in (12). Thus given a solution s_{ijkl} of (8), (9), (11), the dimension of the solution space of the remaining equations for the t_{ijk} is $n\binom{n+1}{2} - n$. □

Using the algorithm contained in Prop. III.1.15 in [2], we will calculate the Cartan characters $s_1, \cdots s_n$ of (6) with respect to the flag

$$0 \subset E_1 \subset \cdots E_{n-1} \subset E_n = E,$$

where $E_j \subset E$ is annihilated by the forms ω^k, $k > j$. If we use $\rho_{ijk} = g_{ip}\pi^p_{jk} + g_{jp}\pi^p_{ik}$ as before, the structure equations (7) become

$$2g^{jk}\rho_{ijk} = g^{jk}\rho_{jki} \tag{15}$$

$$\Psi_{ij} = \rho_{ijk}\omega^k \tag{16}$$

$$d\sigma_i \equiv \sigma_{ij} \wedge \omega^j \quad \mod\{\rho_{ijk}\} \tag{17}$$

$$(-1)^{n-3}\, 2\Theta_i = g^{jk}g^{lm}\rho_{ikl}\omega_{(jm)} - g^{jk}g^{lm}\rho_{lmk}\omega_{(ij)} \tag{18}$$

$$d\Theta_i \equiv 2g^{jk}(\sigma_{ij}w_{(k)} + \sigma_{ji}w_{(k)} - \sigma_{jk}\omega_{(i)}). \tag{19}$$

When $n > 3$, we can inspect the 2-forms in (16), (17) to calculate the characters $s_1, \cdots, s_{n-3}$. For $m < n-2$ we find

$$s_1 + \cdots + s_m = m\binom{n+1}{2} + mn$$

where the first term comes from the ρ_{ijk}, $k \le m$, and the second from the σ_{ik}, $k \le m$. By the same count $s_{n-2} \ge \binom{n+1}{2} + n$, but we must also take into account the extra polar space annihilators that occur as coefficients of $\omega_{(n-1,n)}$ in (18).

To simplify the relation (15), assume that g_{ij} is the identity matrix; later, it will turn out this assumption is not important. To simplify notation in subscripts, let 6 stand for $n-1$ and 7 stand for n, and let a circumflex over repeated indices represent a summation where the index runs over values less than 7. Now solve the relation (15):

$$\rho_{i77} = \begin{cases} \frac{1}{2}\rho_{jji} - \rho_{i\hat{j}\hat{j}}, & i < 7 \\ \rho_{\hat{j}\hat{j}7} - 2\rho_{7\hat{j}\hat{j}}, & i = 7. \end{cases}$$

Using these equations, the coefficient of $\omega_{(67)}$ in Θ_i is congruent to

$$\begin{cases} \rho_{i67} - \rho_{i76}, & i < 6 \\ \rho_{766} + \rho_{667} - 2\rho_{\hat{j}\hat{j}7}, & i = 6 \\ 2\rho_{\hat{j}\hat{j}6} - 2\rho_{666} + \rho_{jj6}, & i = 7 \end{cases}$$

modulo the ρ_{ijk} for $k < 6$. With these extra annihilators, we get

$$s_{n-2} = \binom{n+1}{2} + n + n.$$

From our calculation in the proof of Prop. 3, $s_n = 0$ and

$$s_1 + \cdots + s_{n-1} = \operatorname{codim} E = n\binom{n+1}{2} - n + n^2$$

so $s_{n-1} = 2\binom{n+1}{2}$. Now we compute that

$$s_1 + 2s_2 + 3s_3 + \cdots + ns_n = \binom{n+1}{2}^2 - n^2 - \binom{n+1}{2} + n\binom{n+1}{2} - n.$$

Since the space of integral elements of (6) satisfying the independence condition form a smooth affine bundle over N with rank precisely this number, by Cartan's Test (III.1.11 in [2]) we conclude that our arbitrary integral n-plane E at a point where $g_{ij} = \delta_{ij}$ is ordinary. However $Gl(n)$ has an obvious action on the fibre of N over U, under which any g_{ij} can be taken to the identity matrix. Since the structure equations (7) are $Gl(n)$-invariant, we can conclude that every integral n-plane is ordinary. Thus the system is involutive. In particular, applying the Cartan-Kähler Theorem (III.2.3 in [2]) we conclude

Theorem 5. Through every point in N there exists a smooth analytic integral n-manifold of the system (6) satisfying the independence condition $\Omega \ne 0$. In particular, every 1-jet of a vector field and 1-jet of a metric satisfying $g^{ij}\Gamma^k_{ij} = 0$ is the 1-jet of an analytic soliton metric defined on a small neighbourhood in U.

Note that all local solitons are obtained this way, since by Prop. 3 they will be analytic.

We can now improve on Prop. 3 by noting that involutivity and our calculation above imply that the codimension of the characteristic variety $\Xi^{\mathbb{C}}_E$ is one and the degree is $s_{n-1} = 2\binom{n+1}{2}$ (see Cor. V.3.7 in [2]). Since we proved that $\Xi^{\mathbb{C}}_E$ is contained in the irreducible quadric hypersurface $g^{ij}\xi_i\xi_j = 0$, we deduce that $\Xi_E^{\mathbb{C}}$ must *be* this quadric counted with multiplicity $\binom{n+1}{2}$.

The classical inference that soliton metrics in harmonic coordinates depend on $s_{n-1} = n(n+1)$ arbitrary functions of $n-1$ variables comes from the mechanism in the proof of Cartan-Kähler used to create a sequence of problems to which the Cauchy-Kowalevski theorem applies. After going through the involutivity calculation, one should attempt to find out just what the arbitrary functions are. In fact, the author has calculated that if the vector field X is assumed to be the coordinate vector field $\dfrac{\partial}{\partial x_n}$ then the components g_{ij} of the metric, and their first derivatives in the x_n direction, can be specified along the hyperplane $x_n = 0$ transverse to X; this would account for $n(n+1)$ arbitrary functions along the hyperplane.

Counting identities and the system with fixed metric

When an exterior differential system becomes involutive we know that all its prolongations will also be involutive, and there is a simple way of calculating the Cartan characters of successive prolongations. This in turn enables us to calculate the rank of the bundle of integral n-planes for each prolongation. This gives a count of identities satisfied by solutions, in the following way. In terms of our earlier discussion of involutivity, if we know the rank of the submersion $\mathcal{R}^{(k+1)} \to \mathcal{R}^{(k)}$, we can calculate the codimension of $\mathcal{R}^{(k+1)}$ in the space of "free" $(k+1)$-jets. We will think of this as the number of identities the $(k+1)$-jets of solutions satisfy. Finally, $\mathcal{R}^{(k+1)}$ is the bundle of integral elements of the contact system on $\mathcal{R}^{(k)}$, since a possible tangent plane to an integral manifold passing through a point of $\mathcal{R}^{(k)}$ (i.e. a k-jet of a solution) is a $(k+1)$-jet of a solution.

We are interested in identities that result from imposing the soliton condition in addition to the harmonic coordinate condition, so we will base our comparisons on the latter. The system for a metric g_{ij} with respect to which the coordinates are harmonic lives on $U \times S_+ \times R$ and is generated by the 1-forms

$$\gamma_{ij} = dg_{ij} - \rho_{ijk}\omega^k,$$

where $\rho_{ijk} \equiv g_{ip}d\Gamma^p_{jk} + g_{jp}d\Gamma^p_{ik} \operatorname{mod} \omega^l$ and we have the dependence relation

$$g^{jk}(2\rho_{ijk} - \rho_{jki}) = 0.$$

It is easy to calculate that this system is involutive with characters $s_1 = \ldots = s_{n-1} = \binom{n+1}{2}$, $s_n = \binom{n+1}{2} - n$. These characters will let us count the dimensions of the free k-jets relative to the soliton condition.

In the following table, the entry in the kth row, $k \geq 0$, is the dimension of the bundle of k-jets of solutions over a $(k-1)$-jet solution. To obtain the number of identities at each level we subtract the number in the third column from the sum of the numbers in the first two columns.

	g_{ij} with harmonic coordinates	vector field X	solution to (2)	# of identities
0-jets	$\binom{n+1}{2}$	n	$\binom{n+1}{2}+n$	—
1-jets	$n\binom{n+1}{2}-n$	n^2	$n\binom{n+1}{2}-n+n^2$	—
2-jets	$\binom{n+1}{2}^2-n^2$	$n\binom{n+1}{2}$	$\frac{n(n+1)(n^2+3n-6)}{4}$	$\binom{n+1}{2}+n$
3-jets	$\frac{n^2(n^2-1)(n+4)}{12}$	$n\binom{n+2}{3}$	$\frac{n(n-1)(n^3+7n^2+6n-12}{12}$	$n\binom{n+1}{2}+n^2-n$
4-jets	$\frac{n^2(n^2-1)(n+2)(n+5)}{48}$	$n\binom{n+3}{4}$	$\frac{n^2(n-1)(n+7)(n^2+3n-2)}{48}$	$\binom{n+1}{2}^2+n\binom{n}{2}$

It is easy to see that the 2-jet identities must be

$$X_{ij}+X_{ji}=R_{ij}+\lambda\, g_{ij},$$

which is second-order only in g, together with

$$g^{jk}(X_{ijk}+X_{jik}-X_{jki})=0,$$

which is implied by the second Bianchi identity for the covariant derivative of the Ricci tensor.[2] Moreover, involutivity implies that the identities on the 3-jets will be obtained by differentiating the identities on the 2-jets just once — i.e. no further identities will appear by differentiating twice and equating mixed partials.

Suppose, however, that we only want to know what identities there are that are first-order in X, and we do not care how many derivatives of g are involved. Then what we should do is start with a fixed metric g, and find out what integrability conditions arise from (1), which is now an overdetermined equation for X.

To this end we will define a differential system on the orthonormal frame bundle $\mathcal{F}$ of g, with canonical forms ω^i and Levi-Civita connection forms ϕ^i_j satisfying the structure equations

$$\begin{aligned} d\omega^i &= -\phi^i_j\wedge\omega^j \\ \phi^j_i &= -\phi^i_j \\ d\phi^i_j &= -\phi^i_k\wedge\phi^k_j+\Phi^i_j \\ \Phi^i_j &= \tfrac{1}{2}R^i_{jkl}\omega^k\wedge\omega^l \end{aligned}$$

On $\mathcal{F}\times\mathbb{R}^{\backslash}\times\mathbb{R}^{\binom{\backslash}{\in}}$ define the forms

$$\eta_i=dX_i-(X_j\phi^j_i+(R_{ij}+\lambda\delta_{ij}+A_{ij})\omega^j),\qquad A_{ji}=-A_{ij}$$

Then $-d\eta_i\equiv X_j\Phi^j_i+(R_{ijk}\omega^k+\pi_{ij})\wedge w^j \mod \eta_l$, where

$$\pi_{ij}=dA_{ij}-A_{ik}\phi^k_i-A_{kj}\phi^k_j$$

and $R_{ijk}=\nabla_k R_{ij}$ is the covariant derivative of the Ricci tensor.

We are looking for n-manifolds along which $\Omega\neq 0$ and the 1-forms η_i vanish. The Cartan system for this Pfaffian system is $\{\omega^i,\eta_i,\pi_{ij}\}$. When we try to find integral n-planes we can ignore the 1-forms ϕ^i_j in our coframe which do not appear in this system. (Strictly speaking, the Pfaffian system drops to the quotient $\mathcal{F}/\mathcal{O}(\backslash)$, but

[2] If we assume that X is a divergence, i.e. $f_{ij}=R_{ij}+\lambda\, g_{ij}$ for some function f, then this identity simplifies to

$$2f_jR^j_i+\nabla_iR=0,$$

which would determine f up to a constant if the Ricci curvature were non-degenerate.

we will continue to work with forms on $\mathcal{F}$.) It turns out that there is a unique integral n-plane at each point, given by

$$\pi_{ij}|_E = (R_{kij} - R_{kji} - X_p R^p_{kij})\omega^k.$$

Thus we should add the corresponding 1-forms

$$\theta_{ij} = \pi_{ij} + (X_p R^p_{kij} - R_{kij} + R_{kji})\omega^k$$

to our system. Now the θ_{ij} must be closed modulo the system $\{\eta_i, \theta_{ij}\}$; otherwise, there will be no integral n-planes satisfying $\Omega|_E \neq 0$. In this way, calculating $d\theta_{ij}$ gives us integrability conditions.

The expression of this integrability condition has the following interesting geometric interpretation. Let Θ be the sheaf of germs of Killing fields on a Riemannian manifold, and let D_0 be the first-order linear differential operator taking X to the symmetric bilinear form $X_{ij} + X_{ji}$. In the case where the metric has constant curvature Calabi [3] defined a fine resolution of Θ

$$0 \to \Theta \hookrightarrow \Phi^0 \xrightarrow{D_0} \Phi^1 \xrightarrow{D_1} \Phi^2 \xrightarrow{D_2} \dots$$

where Φ^0 is the sheaf of sections of TM, Φ^1 is the sheaf of symmetric bilinear forms, and Φ^k is the sheaf of sections of the kernel of the map

$$\Lambda^2 T^*M \otimes \Lambda^k T^*M \to T^*M \otimes \Lambda^{k+1} T^*M$$

which anti-symmetrizes on the last $k+1$ factors. (In particular the symmetries of the Riemann curvature tensor imply that it lies in Φ^2.) Gasqui and Goldschmidt have also achieved resolutions of Θ in the case of locally symmetric [5] and conformally flat spaces [6].

For our purposes, define $D_1 : \Phi^1 \to \Phi^2$ as follows: if $s \in \Phi^1$ has components s_{ij}, let

$$D_1(s)_{ijkl} = s_{iljk} - s_{jlik} - s_{ikjl} + s_{jkil} + s_{ip}R^p_{jkl} - s_{jp}R^p_{ikl},$$

where s_{ijkl} are the components of the second covariant derivative of s. (It is clear that $D_1(s)$ is a section of $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$; to see that $D_1(s) \in \Phi^2$, calculate

$$\begin{aligned} D_1(s)_{ijkl}\,\omega^j \wedge \omega^k \wedge \omega^l &= (s_{iljk} - s_{ikjl} - s_{jp}R^p_{ikl})\,\omega^j \wedge \omega^k \wedge \omega^l \\ &= (s_{ijkl} - s_{ijlk} - (s_{jp}R^p_{ikl} + s_{ip}R^p_{jkl}))\,\omega^j \wedge \omega^k \wedge \omega^l = 0 \end{aligned}$$

using the first Bianchi identity, $R^p_{jkl}\omega^j \wedge \omega^k \wedge \omega^l = 0$.) This D_1 coincides with Calabi's operator in the constant curvature case. $D_1 \circ D_0$ is not zero in general, but it is only first-order in X:

$$\begin{aligned} D_1 \circ D_0(X)_{ijkl} = 2(&X_p(R^p_{lijk} - R^p_{kijl}) + X_{pk}R^p_{lij} - X_{pl}R^p_{kij}) \\ &+ (X_{ip} - X_{pi})R^p_{jlk} + (X_{pj} - X_{jp})R^p_{ilk}. \end{aligned}$$

Now, the soliton condition (1) can be written as

$$D_0(X) = 2\,\mathrm{Rc} + \lambda\, g$$

and the integrability condition for the exterior differential system with g fixed is

$$D_1(D_0(X)) = D_1(2\,\mathrm{Rc} + \lambda\, g)$$

which is first-order in X but fourth-order in g. The author has calculated that the soliton system for 1-jets of X and 4-jets of g satisfying the above two conditions in involutive. Hence these exhaust the identities implied by the soliton condition that are first-order in X.

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